On semisimple Hopf actions

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The Road goes ever on and on
Down from the door where it began.
Now far ahead the Road has gone,
And I must follow, if I can,
Pursuing it with eager feet,
Until it joins some larger way
Where many paths and errands meet.
And whither then? I cannot say.

J. R. R. Tolkien

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Abstract

Suppose that H is a semisimple Hopf algebra over a field F acting on an algebra A. We study the following question: when does this action factors through a group action?

In [18], for H semisimple and F algebraically closed, the authors showed that if A is a commutative domain, then the action factors through a group action. Cuadra, Etingof and Walton, in [13], showed that any semisimple Hopf action on the Weyl algebra $A_n(F)$ must factor through a group action.

In this work, with F algebraically closed of characteristic zero, we show that any action of a semisimple Hopf algebra H on an enveloping algebra of a finite-dimensional Lie algebra or on an iterated Ore extension of derivation type $A = F[x_1][x_2; d_2][\cdots][x_n; d_n]$ factors through a group action. In order to do this, we consider inner faithful Hopf actions and use a reduction step, which basically consist of passing from algebras in characteristic zero to algebras in positive characteristic.

Next, we present actions of semisimple Hopf algebras over an algebraically closed field of characteristic zero which do not factor through group actions. For this, we construct a class of Hopf algebras H_{2n^2}, which are not group algebras, and establish conditions to define inner faithful actions of those Hopf algebras on the quantum polynomial algebras. In this way, we give examples of algebras where the question stated at the beginning is answered negatively.

Resumo

Seja H uma álgebra de Hopf semi-simples sobre um corpo F que age numa álgebra A. Estudamos a seguinte questão: Quando esta acção factoriza-se por uma acção de grupo?

Em [18], para o caso de H semi-simples e F algebricamente fechado, os autores mostraram que se A é um domínio comutativo, então a acção factoriza-se por uma acção de grupo. Cuadra, Etingof and Walton, em [13], mostraram que tal também acontece para qualquer acção de uma álgebra de Hopf semi-simples na álgebra de Weyl $A_n(F)$.

Neste trabalho, dado F um corpo algebricamente fechado de característica zero, mostramos que qualquer acção de uma álgebra de Hopf semisimples numa álgebra envolvente de uma álgebra de Lie de dimensão finita ou numa extensão de Ore iterada do tipo de derivação $A = F[x_1][x_2; d_2][\cdots][x_n; d_n]$ factoriza-se por uma acção de grupo. Para tal, consideramos acções de ágebras de Hopf fielmente internas e usamos uma redução, que basicamente consiste em passar de álgebras em característica zero para álgebras em característica positiva.

Apresentamos também acções de álgebras de Hopf semi-simples sobre um corpo algebricamente fechado de característica zero que não se factoriza por acções de grupo. Para isso, construímos uma classe de álgebras de Hopf, H_{2n^2}, que não são álgebras de grupo, e estabelecemos condições para definir acções fielmente internas de tais álgebras de Hopf em álgebras polinomiais quânticas. Obtemos assim exemplos de álgebras onde a questão colocada é respondida negativamente.

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Chapter 1

Introduction

1.1 Overview

The concept of Hopf algebras arose in the 1940's in the context of algebraic topology and cohomology in relation to the work of Heinz Hopf [22]. Basically, two different strands, one coming from the theory of algebraic groups and the other from algebraic topology, are responsible for, historically speaking, introducing the term "Hopf algebra" in the literature. Both of them appeared during the $1950'$s. It was only in the $1960'$s and $1970'$s, with the works of Larson, Radford, Sweedler, Taft, and Wilson, among others, that a general theory of Hopf algebras was developed, with the first book on the subject written by Sweedler and published in 1969 [49]. For more information on the beginning history of Hopf algebras, see the survey of Andruskiewitsch and Ferrer [4]. In the $1980'$s, most specifically with the floodgates opened by Drinfel'd [16], connections between Hopf algebras and theoretical physics were discovered, the so called quantum groups, as well as connections with quantum topology, noncommutative geometry, knot theory etc. For more on quantum groups, see [26] and [36].

A standard approach to introduce Hopf algebras is through group algebras. Briefly, a Hopf algebra H is a vector space over a field F with an algebra and a coalgebra structure along with a linear map $S : H \to H$, called antipode, satisfying certain compatibility conditions. The most standard examples of Hopf algebras are group algebras $F[G]$, for G a group, and in this case, the antipode S is defined by the inverse operation of the group G, $S(g) = g^{-1}$ for all $g \in G$. In a certain sense, the notion of Hopf algebras generalizes the notion of groups. Now, in group theory, one can always look to groups in a different way, namely by its actions on other mathematical objects, instead of looking at the multiplication structure within the group. Geometers use group actions on geometric objects to find out informations about the structure of the object, while group theorists use group actions on a set to study the structure of the group itself. Often, the actions of the group are symmetries of the object.

Roughly speaking, several results for groups were "transfered" from group theory to the Hopf algebraic context. Perhaps the most striking of them are Maschke's theorem [Theorem 2.2.36] and Lagrange's theorem for Hopf algebras [Corollary 2.2.44]. Also, in a different direction, there has been done a lot of work on Hopf algebra actions, allowing, for instance in noncommutative geometry, to codify symmetries of noncommutative spaces that cannot be codified by a group (examples of this phenomenon are quantum groups).

Suppose that H is a finite-dimensional Hopf algebra over a field F acting on an algebra A. Ff I is a Hopf ideal such that $I \cdot A = 0$, we say that the action *factors through* a quotient Hopf algebra H/I. One says that the action *factors through a group action* if there exists a Hopf ideal I of H, with $I \cdot A = 0$, such that $H/I \cong F[G]$ as Hopf algebra for some group G. In this last scenario, the Hopf action can be seen, in a certain sense, as a group action. Suppose that H is a semisimple Hopf algebra over a field F acting on an algebra A. When does this action factors through a group action?

The first general result appeared in [18]. Assuming that H is semisimple and F is algebraically closed, the authors showed that if A is a commutative domain, then the action factors thorough a group action in this setting. Cuadra, Etingof and Walton, in [13], showed that that is also the case for the Weyl algebra $A = A_n(F)$, i.e., they showed that any semisimple Hopf action on the Weyl algebra $A_n(F)$ must factor through a group action.

In this thesis, by using and analyzing the results obtained by Cuadra, Etingof and Walton, we prove that the action factors through a group action for the case where A is an enveloping algebra of a finite-dimensional Lie algebra or an iterated Ore extension of derivation type $A = F[x_1; d_1][x_2; d_2][\cdots][x_n; d_n]$, and H a semisimple Hopf algebra over an algebraically closed field F of characteristic zero, i.e., the action must factor through a group action. Also, we shall give examples of algebras where there exists an action of a finite-dimensional Hopf algebra that does not factor through a group action. To do this, we present a construction of a family of semisimple Hopf algebras H_{2n^2}, for n a positive integer, and we give conditions to define actions of such Hopf algebras on quantum polynomial algebras.

1.2 Thesis organization

The thesis is organized as follows. In chapter 2, we shall review some basic terminology, notation, and results dealing with finite-dimensional Hopf algebras, including results on semisimple Hopf algebras and the Nichols-Zoeller Theorem [Theorem 2.2.43].

In chapter 3, we will analyze the main result of the paper [13] to show that any action of a semisimple Hopf algebra H on an enveloping algebra of a finite-dimensional Lie algebra or on an iterated Ore extension of derivation type $A = F[x_1; d_1][x_2; d_2][\cdots][x_n; d_n]$ factors through a group action. In order to do this, we will consider inner faithful Hopf actions and use a reduction step, which basically consist of passing from algebras in characteristic zero to algebras in positive characteristic by using a subring R of the ground field F, which is generated by all structure constants of H and the action on A, and by passing to a finite field $R/\mathfrak{m}$, for $\mathfrak{m}$ a maximal ideal of R.

In chapter 4, we will present actions of semisimple Hopf algebras over an algebraically closed field of characteristic zero which do not factor through group actions. In order to do that, we will construct a class of Hopf algebras H_{2n^2}, which are not group algebras, and establish conditions to define inner faithful actions of those Hopf algebras on the quantum polynomial algebras. In a recent paper, [19], P. Etingof and C. Walton say that there is *no finite quantum symmetry* when the action of any finite-dimensional Hopf algebra factors through a group action. In this way, we give examples of algebras where there is quantum symmetry.

Chapter 2

Preliminares

Throughout this work, although much of what will be presented here can be done over any commutative ring, we let F be a field. Unless it is said otherwise, all the tensor products, vector spaces, algebras and Hopf algebras are taken over the same ground field F. Moreover, unless stated otherwise, by *algebra*, we mean an unital associative F-algebra.

The theory of Hopf algebras, especially after the works which have been done during the last decades, can be considered a well-established branch of algebra with standard textbooks on this subject [1], [17], [39], [42], [46], [49]. Nevertheless, in this chapter, we shall collect some basic facts on Hopf algebras, on their actions on algebras, and some general results. The notation and convention introduced in this section will be used in all the thesis.

2.1 Basic definitions

In this section, we introduce the basic definitions of the theory of coalgebras and discuss the duality between algebras and coalgebras.

2.1.1 Coalgebras

We begin with some basic assumptions. Namely, we start expressing the associativity and unit properties of an algebra in terms of diagrams and maps in order to dualize them to obtain the definition of an F-Coalgebra.

Definition 2.1.1. *An F-algebra (A, μ, η) is an F-vector space A together with F-linear maps $\mu : A \otimes A \to A$ and $\eta : F \to A$ such that the following diagrams are commutative:*

$$
\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "id_A\otimes\mu"] \arrow[d, "\mu\otimes id_A"'] & A\otimes A \arrow[d, "\mu"] \\
A\otimes A \arrow[r, "\mu"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
& A\otimes A \arrow[dd, "\mu"] & \\
F\otimes A \arrow[ur, "\eta\otimes id_A"] & & A\otimes F \arrow[ul, "id_A\otimes\eta"'] \\
& A \arrow[ul, "\simeq"] \arrow[ur, "\simeq"'] &
\end{tikzcd}
$$

The isomorphisms in the second diagram are given by scalar multiplication. The first commutative diagram expresses the associativity property of A. The second diagram expresses the unity property of A. We denote $\eta(1_F) = 1_A$.

The linear map $\tau_{V,W} : V \otimes W \to W \otimes V$ defined by $\tau_{V,W}(v \otimes w) = w \otimes v$ is called the *twist map*. We note that an algebra (A, μ, η) is commutative if and only if $\mu \circ \tau_{A,A} = \mu$.

Hereafter, when we refer to an algebra (A, μ, η), we will purposely omit the structure maps μ and η and refer to the algebra simply by A.

Now, in order to define Hopf algebras, we dualize the definition of an algebra by reversing the arrows of the diagrams replacing the F-linear maps appropriately.

Definition 2.1.2. *A coalgebra (C, Δ, ϵ) is an F-vector space C together with F-linear maps $\Delta : C \to C \otimes C$ and $\epsilon : C \to F$ such that the following diagrams are commutative:*

$$
\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C\otimes C \arrow[d, "id_C\otimes\Delta"] \\
C\otimes C \arrow[r, "\Delta\otimes id_C"'] & C\otimes C\otimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
& C \arrow[dd, "\Delta"] & \\
F\otimes C \arrow[ur, "\cong"] & & C\otimes F \arrow[ul, "\cong"'] \\
& C\otimes C. \arrow[ul, "\epsilon\otimes id_C"] \arrow[ur, "id_C\otimes\epsilon"'] &
\end{tikzcd}
$$

The first diagram is referred to as *coassociativity axiom* with Δ being called *comultiplication*. And the second diagram is referred to as *counity property* with ϵ being called *counity*. In order to perform calculations on coalgebras, we use the *Sweedler notation* for the comultiplication. For an element $c \in C$ we write

$$\Delta(c) = \sum c_1 \otimes c_2,$$

where the summation is understood as a finite sum and the notation is just symbolic in the sense that the terms of the sum do not represent elements of C. This notation, while it may be an abuse of notation, is very useful in calculations. For instance, using the coassociativity axiom, and this notation, we have then

$$(\Delta \otimes id)\Delta(c) = \sum c_{1_1} \otimes c_{1_2} \otimes c_2 = \sum c_1 \otimes c_{2_1} \otimes c_{2_2} = (id \otimes \Delta)\Delta(c),$$

for all $c \in C$. Hence, we just denote

$$(\Delta \otimes id)\Delta(c) = (id \otimes \Delta)\Delta(c) = \sum c_1 \otimes c_2 \otimes c_3.$$

We write $(id \otimes \Delta) \circ \Delta = \Delta_2 = (\Delta \otimes id) \circ \Delta$. The coassociativity axiom, then, allow us to write

$$\Delta_n(c) = \sum c_1 \otimes \cdots \otimes c_{n+1},$$

where $\Delta_n(c)$ is obtained by applying the coassociativity axiom n times.

Also, in this notation, for any $c \in C$, the counity axiom can be expressed as

$$\sum \epsilon(c_1)c_2 = c = \sum c_1\epsilon(c_2).$$

The reader is refereed to [42, Chapters 2,3,4] for a more solid introduction to the theory of coalgebras including the use of Sweedler notation.

From now on, when we refer to a coalgebra (C, Δ, ϵ) we will omit the structures maps Δ and ϵ and just refer to the coalgebra as C.

Definition 2.1.3. *Let C be a coalgebra. A vector subspace D of C is called a subcoalgebra if $\Delta(D) \subseteq D \otimes D$.*

We say that a coalgebra C is *cocommutative* if $\Delta(c) = (\tau_{C,C} \circ \Delta)(c)$ for all $c \in C$.

Example 2.1.4. *Given two coalgebras C and D, the tensor product $C \otimes D$ is a coalgebra with comultiplication defined as $\Delta_{C\otimes D} = (id \otimes \tau_{C,D} \otimes id) \circ (\Delta_C \otimes \Delta_D)$ and counity $\epsilon_{C\otimes D} = \epsilon_C \otimes \epsilon_D$, where $\tau_{C,D} : C \otimes D \to D \otimes C$ is the twist map.*

The following two examples can be found in [17, Examples 1 and 4 in Section 4.3].

Example 2.1.5. *Let G be any group and let $D = F[G]$ be its group algebra. D becomes a coalgebra if we define $\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$, for all $g \in G$.*

More general, let S be any non-empty set. Let $F[S]$ be the F-vector space with canonical basis S. $F[S]$ becomes a coalgebra by defining $\Delta(s) = s \otimes s$ and $\epsilon(s) = 1$, for all $s \in S$.

Example 2.1.6. *Let $\mathfrak{g}$ be any Lie algebra over F and let $B = U(\mathfrak{g})$ its universal enveloping algebra. By defining $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $\epsilon(x) = 0$, B becomes a coalgebra.*

In any coalgebra C, elements whose comultiplication is as in the first above example are sort of special and important. We say that an element $c \in C$ is a *group-like* element if $\Delta(c) = c \otimes c$. For a group-like element $c \neq 0$, the counity property ensures that $\epsilon(c) = 1$. If we set $\mathrm{G}(C)$ the set of all group-like elements of C, then $F[\mathrm{G}(C)]$ is a subcoalgebra of C.

Lemma 2.1.7 ([42, Lemma 2.1.12])**.** *Let C be a coalgebra and assume $\mathrm{G}(C)$ is non-empty. Then $\mathrm{G}(C) \subseteq C$ is a linearly independent subset of C.*

Let $(C, \Delta_C, \epsilon_C)$ and $(D, \Delta_D, \epsilon_D)$ be two coalgebras. An F-linear map $f : C \to D$ is called a *coalgebra homomorphism* if the diagrams

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\ C \otimes C \arrow[r, "f \otimes f"'] & D \otimes D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\epsilon_C"'] & & D \arrow[dl, "\epsilon_D"] \\ & k & \end{tikzcd}$$

commute. That is to say, in terms of the Sweedler notation, that, for all $c \in C$, $\Delta_D(f(c)) = \sum f(c)_1 \otimes f(c)_2 = \sum f(c_1) \otimes f(c_2) = (f \otimes f)(\Delta(c))$ and $(\epsilon_D \circ f)(c) = \epsilon_C(c)$.

The ground field F has a natural structure of coalgebra with $\Delta(1) = 1 \otimes 1$ and $\epsilon(1) = 1$. Moreover, for any coalgebra C, $\epsilon : C \to F$ is a morphism of coalgebras.

Definition 2.1.8. *A subspace $I \subseteq C$ is called a left coideal (right coideal) if $\Delta(I) \subseteq C \otimes I$ (respectively $\Delta(I) \subseteq I \otimes C$). I is a coideal if $\Delta(I) \subseteq I \otimes C + C \otimes I$ and if $\epsilon(I) = 0$.*

Proposition 2.1.9 ([17, Proposition 1.4.9])**.** *Suppose that $f : C \to D$ is a homomorphism of coalgebras. Then $\mathrm{Im}(f)$ is a subcoalgebra of D and $\mathrm{Ker}(f)$ is a coideal in C.*

We shall denote the coideal $\mathrm{Ker}(\epsilon)$ by C^+. The last result of this subsection, which can be found in [17, Theorem 1.4.10] or [42, Theorem 2.1.21], is the *homomorphism theorem* for coalgebras.

Theorem 2.1.10. *Let C be a coalgebra, I a coideal and $\pi : C \to C/I$ the canonical projection of vector spaces. Then:*

i) There exists a unique coalgebra structure on C/I such that π is a morphism of coalgebras;

ii) If $f : C \to D$ is a morphism of coalgebras with $I \subseteq \mathrm{Ker}(f)$, then there exists a unique morphism of coalgebras $\overline{f} : C/I \to D$ with $\overline{f} \circ \pi = f$.

2.1.2 Duality between algebras and coalgebras

In this subsection, we present a close relationship between algebras and coalgebras. For this, we will look to their dual spaces.

For any vector space V, let $V^* = \mathrm{Hom}_F(V, F)$ denote the linear dual of V. For any linear map $\phi : V \to W$ between vector spaces, we can consider the *transpose* of ϕ, $\phi^* : W^* \to V^*$, which is given by $\phi^*(f)(v) = f(\phi(v))$, for all $f \in W^*$, $v \in V$. Also, we have a linear inclusion $V^* \otimes W^* \hookrightarrow (V \otimes W)^*$ given by $(f \otimes g)(v \otimes w) = f(v)g(w)$, for all $v \in V$, $w \in W$, $f \in V^*$, and $g \in W^*$. Such a map is an isomorphism of vector spaces whenever V and W are finite-dimensional.

For a coalgebra C, we can consider the transpose map Δ^* and ϵ^*. What happens when we consider such transpose maps is stated in the following lemma, where by Δ^* we mean the composition $C^* \otimes C^* \hookrightarrow (C \otimes C)^* \xrightarrow{\Delta^*} C^*$.

Lemma 2.1.11 ([39, Lemma 1.2.2]). *If C is a coalgebra, then C^* is an algebra with product Δ^* and unity ϵ^*. If C is cocommutative, then C^* is commutative.*

In a more general setting, given an algebra A and a coalgebra C, the space of all linear maps $\mathrm{Hom}_F(C, A)$ becomes an algebra via the convolution product: $f * g = \mu \circ (f \otimes g) \circ \Delta$, for all $f, g \in \mathrm{Hom}(C, A)$, while the identity is given by $\eta \circ \epsilon$. And in the above lemma, we are just taking $A = F$.

Now, if we start with a finite-dimensional algebra A, we have then that $A^* \otimes A^* \cong (A \otimes A)^*$. This allows us to define maps $\Delta_{A^*} : A^* \to A^* \otimes A^*$ as the transpose of the multiplication map and the unity $\epsilon_{A^*} : A^* \to F$. Explicitly, $\Delta_{A^*}(f) = \sum g_i \otimes h_i$ for any $(g_i, h_i) \in A^*$ with the property of $f(ab) = \sum g_i(a)h_i(b)$, for any $a, b \in A$, and $\epsilon_{A^*}(f) = f(1_A)$, for all $f \in A^*$.

Proposition 2.1.12 ([17, Proposition 1.3.9]). *If A is a finite-dimensional algebra, then $(A^*, \Delta_{A^*}, \epsilon_{A^*})$ is a coalgebra.*

Remark 2.1.13. *Let A be a finite-dimensional algebra. Let $\{e_1, \cdots, e_n\}$ be a basis for A and $\{f_1, \ldots, f_n\}$ be a dual basis for A^*, i.e., $f_j(e_i) = \delta_{i,j}$ for all $1 \leq i, j \leq n$, where δ is the*

Kronecker symbol. For any $a, b \in A$, we can write $a = \sum_{l=1}^{n} \alpha_l e_l$ and $b = \sum_{p=1}^{n} \beta_p e_p$, for $\alpha_l, \beta_p \in F$. Note that, for any $g \in A^*$, we have

$$\begin{aligned}\sum_{i,j=1}^{n} g(e_ie_j)f_i(a)f_j(b) &= \sum_{i,j=1}^{n}\sum_{l,p=1}^{n} g(e_ie_j)\alpha_l\beta_p f_i(e_l)f_j(e_p)\\ &= \sum_{i,j=1}^{n} g(e_ie_j)\alpha_i\beta_j\\ &= g(ab).\end{aligned}$$

Therefore, $\Delta_{A^*}(g) = \sum_{i,j=1}^{n} g(e_ie_j)f_i \otimes f_j$, for all $g \in A^*$.

Also, we have the following result saying something about homomorphisms of algebras and coalgebra and their duals.

Proposition 2.1.14 ([17, Proposition 1.3.12])**.** *Let C and D be coalgebras. Also, let A and B be finite-dimensional algebras. Thus*

i) If $f : C \to D$ is a homomorphism of coalgebras, then $f^* : D^* \to C^*$ is a homomorphism of algebras;

ii) If $f : A \to B$ is a homomorphism of algebras, then $f^* : B^* \to A^*$ is a homomorphism of coalgebras.

The problem when A is not finite-dimensional is that $A^* \otimes A^*$ is a proper subspace of $(A \otimes A)^*$ and thus the image of the transpose map $\mu^*(A^*)$ may not lie in $A^* \otimes A^*$. In this case, we have to consider a subspace of A^* called *finite dual* of A. The finite dual of A is $A^\circ = \{f \in A^* | f(I) = 0, \text{ for some ideal } I \text{ of } A \text{ such that } \dim(A/I) < \infty\}$. With this space, we have a more general result.

Proposition 2.1.15 ([39, Proposition 1.2.4])**.** *If A is an algebra, then A° is a coalgebra, with comultiplication $\Delta = \mu^*$ and counity $\epsilon = \eta^*$. If A is commutative, then A° is cocommutative.*

The finite dual is the largest subspace V of A^* such that $\mu^*(A) \subseteq V \otimes V$. Moreover, if A is finite-dimensional, then $A^\circ = A^*$.

2.2 Bialgebras and Hopf algebras

In this section, we present the notion of *Hopf algebras*, which is, roughly speaking, both an algebra and a coalgebra with a certain compatibility condition. We recall that for any algebra A and B, both $A \otimes B$ and the ground field F can be regarded as algebras. Also, for any coalgebras C and D, both $C \otimes D$ and the ground field F can be regarded as coalgebras.

2.2.1 Bialgebras

Before we properly introduce the concept of a Hopf algebras, we start with the definition of a *bialgebra*. We begin with a proposition which shows a natural compatibility between an algebra structure and a coalgebra structure.

Proposition 2.2.1 ([42, Lemma 5.1.1])**.** *Let (H, μ, η) be an algebra and (H, Δ, ϵ) be a coalgebra. Given the algebra and coalgebra structures on $H \otimes H$ and on F, the following are equivalent:*

i) μ and η are homomorphisms of coalgebras;

ii) Δ and ϵ are homomorphisms of algebras.

Definition 2.2.2. *An F-bialgebra is a tuple $(H, \Delta, \epsilon, \mu, \eta)$, where (H, μ, η) is an algebra and (H, Δ, ϵ) is a coalgebra, such that either of the conditions of Proposition 2.2.1 are satisfied.*

Using the Sweedler notation, to say that $(H, \Delta, \epsilon, \mu, \eta)$ is a bialgebra means that, for all $a, b \in H$ and for $1_H = \eta(1_F)$, $\Delta(ab) = \sum a_1 b_1 \otimes a_2 b_2$, $\Delta(1_H) = 1_H \otimes 1_H$, $\epsilon(ab) = \epsilon(a)\epsilon(b)$, and $\epsilon(1_H) = 1_F$.

Just as we did with algebras and coalgebras, from now on, when we refer to a bialgebra $(H, \Delta, \epsilon, \mu, \eta)$ we will be omitting the structure maps Δ, ϵ, μ and η, and refer to it just as H.

Example 2.2.3. *The ground field F with the natural structure of coalgebra presented before is a bialgebra.*

Again,the following two examples can be found in [17], chapter 4, section 4.3, examples 1) and 4) respectively.

Example 2.2.4 ([17, c. 4, s. 4.3, Example 1)])**.** *Let G be any group. The group algebra $F[G]$ is a bialgebra with the coalgebra structure as in Example 2.1.5.*

Example 2.2.5 ([17, c. 4, s. 4.3, Example 4)]). *Let $\mathfrak{g}$ be a Lie algebra and let $U(\mathfrak{g})$ be its universal enveloping algebra. With the structure of colagebra presented in Example 2.1.6, $U(\mathfrak{g})$ is a bialgebra.*

For bialgebras H and K, a linear map $f : H \to K$ is called *bialgebra homomorphism* if it is both an algebra homomorphism and a coalgebra homomorphism.

The duality between algebras and coalgebras established in the Subsection 2.1.2 is now useful to construct new bialgebras from a given finite-dimensional bialgebra H as presented in the following proposition.

Proposition 2.2.6 ([17, Propostition 4.1.6]). *Let H be a finite-dimensional bialgebra. Then H^*, with the dual structures presented in subsection 2.1.2, is a bialgebra.*

Definition 2.2.7. *Let H be a bialgebra. A subspace $K \subseteq H$ is called a sub-bialgebra if K is simultaneously a sub-algebra and a sub-coalgebra.*

Let H_1 and H_2 be two bialgebras, a map $f : H_1 \to H_2$ is called a *homomorphism of bialgebras* if f is simultaneously a homomorphism of algebras and a homomorphism of coalgebras. A subspace $I \subseteq H_1$ is called a *bi-ideal* if I is both an ideal and a coideal. With this setting, the quotient space H_1/I naturally has a structure of bialgebra, and the projection map $\pi : H_1 \to H_1/I$ is a bialgebra homomorphism [17, Proposition 4.1.8].

2.2.2 Hopf algebras

Finally, we are ready to present the definition of Hopf algebras. A Hopf Algebra H, roughly speaking, is a bialgebra with an anti-homomorphism of algebras $S : H \to H$ satisfying certain conditions.

Before we give the definition, we recall that if C is a coalgebra and A is an algebra, then $\mathrm{Hom}_F(C, A)$ is an algebra with the convolution product $f * g = \mu \circ (f \otimes g) \circ \Delta$ for all $f, g \in \mathrm{Hom}_F(C, A)$, and unity $\eta \circ \epsilon$.

Now, let H be a bialgebra. Since H is an algebra and a coalgebra, $\mathrm{Hom}_F(H, H)$ is an algebra with the convolution product. Moreover, the identity map $id_H : H \to H$ is an element of $\mathrm{Hom}_F(H, H)$. We may consider whether or not id_H is an invertible element, with respect to the convolution product, in $\mathrm{Hom}_F(H, H)$. When it is an invertible element, we have a Hopf algebra.

Definition 2.2.8. *A Hopf algebra is a bialgebra such that the identity map id_H has a convolution inverse S in* $\mathrm{Hom}_F(H, H)$.

Usually, S is called an antipode of H. Also, to say that S is the inverse element of id_H in $\mathrm{Hom}_F(H, H)$ with respect to the convolution product just means that, for any $h \in H$, we must have:

$$\sum S(h_1)h_2 = \epsilon(h)1_H = \sum h_1 S(h_2),$$

or, equivalently, S is an antipode of H if S satisfies $(S\otimes id)\Delta(h) = \epsilon(h)1_H = (id\otimes S)\Delta(h)$, for all $h \in H$.

Before we give some examples, we will present some properties of the antipode S.

Proposition 2.2.9 ([17, Proposition 4.2.6])**.** *Let H be a Hopf algebra with antipode S. Then:*

i) $S(hg) = S(g)S(h)$, *for all* $g, h \in H$*;*

ii) $S(1_H) = 1_H$*;*

iii) $\Delta(S(h)) = \sum S(h_2) \otimes S(h_1)$, *for all* $h \in H$*;*

iv) $\epsilon(S(h)) = \epsilon(h)$, *for all* $h \in H$.

This result shows that S is an algebra anti-homomorphism and a coalgebra anti-homomorphism.

For Hopf algebras H and K, with antipodes S_H and S_K respectively, it would be expected that a bialgebra homomorphism $f : H \to K$ would be *Hopf algebra homomorphism* if, in addition, $S_K \circ f = f \circ S_H$. But, in fact, this is a consequence. If $f : H \to K$ is a bialgebra homomorphism, then $S_K \circ f = f \circ S_H$ (see [17, Proposition 4.2.5]).

If H is a Hopf algebra with antipode S, then a subspace K of H is called a *Hopf subalgebra* of H if K is a sub-bialgebra of H and $S(K) \subseteq K$. Also, in this Hopf algebra setting, we can construct the quotient Hopf algebra, but first we need to define what a *Hopf ideal* is.

Definition 2.2.10. *A subspace $I \subseteq H$ is called a Hopf ideal if I is a bi-ideal of H and $S(I) \subseteq I$.*

For a Hopf ideal I, we know, from the last subsection, that the quotient space H/I is a bialgebra. But the condition $S(I) \subseteq I$ allows us to define a Hopf algebra structure on the quotient, where we set the antipode $\overline{S} : H/I \to H/I$ given by $\overline{S}(h) = S(h) + I$, for all $h \in H$.

Remark 2.2.11. *If H is a Hopf algebra, then the set $G(H)$ of group-like elements of H is a group with the multiplication inherited by the one on H. What guarantees this is that Δ is an algebra homomorphism, the properties of the antipode S and the fact that S is an anti-homomorphism of coalgebras. Note that $1_H \in G(H)$ and that $S(g) = g^{-1}$ for all $g \in G(H)$.*

Lemma 2.2.12. *Let H be a Hopf algebra and I be a Hopf ideal of H. Then every Hopf ideal B of the quotient Hopf algebra H/I is given by J/I for some Hopf ideal J of H containing I.*

Proof. Let I be a Hopf ideal of H and $\pi : H \to H/I$ be the projection map, which is a Hopf algebra homomorphism. Let B be a Hopf ideal of H/I and set $J = \pi^{-1}(B) = \{h \in H \mid h+I \in B\}$. Note that J is an ideal of H, $I \subseteq J$, $\epsilon(J) = 0$, and $S(J) \subseteq J$. In order to prove that J is a Hopf ideal of H it remains to prove that $\Delta(J) \subseteq J \otimes H + H \otimes J$.

Since $\mathrm{Ker}(\pi \otimes \pi) = \mathrm{Ker}(\pi) \otimes H + H \otimes \mathrm{Ker}(\pi)$ [17, Lemma 1.4.8] and $(\pi \otimes \pi)\Delta(J) = 0$, it follows that $\Delta(J) \subseteq \mathrm{Ker}(\pi \otimes \pi) = \mathrm{Ker}(\pi) \otimes H + H \otimes \mathrm{Ker}(\pi) = I \otimes H + H \otimes I$, since $I = \mathrm{Ker}(\pi)$. But $I \subseteq J$. Therefore, $\Delta(J) \subseteq J \otimes H + H \otimes J$ and thus J is a coideal of H.

Clearly, $J/I = B$. □

Lemma 2.2.13. *Let J be a Hopf ideal of a Hopf algebra H which contains a Hopf subalgebra R. Then $I = J \cap R$ is a Hopf ideal of R.*

Proof. Clearly, I is an ideal of R and $S(I) \subseteq I$. So, it remains to prove that I is a coideal of R. Consider the restriction of the projection map $\pi_{|_R} : R \to H/J$. Well, $\pi_{|_R}$ is certainly a coalgebra map and $\mathrm{Ker}(\pi_{|_R}) = I$. Therefore, I is a coideal of R. Hence, I is a Hopf ideal of R. □

Also, a key property for finite-dimensional Hopf algebras is that their linear duals are again Hopf algebras.

Proposition 2.2.14 ([17, Proposition 4.2.11])**.** *Let H be a finite-dimensional Hopf algebra. Then H^* is a Hopf algebra with antipode given by the transpose $S^* : H^* \to H^*$.*

The next proposition will be useful in the next chapter. It answers the question of whether a sub-bialgebra of a Hopf algebra H is itself a Hopf algebra.

Proposition 2.2.15 ([42, Proposition 7.6.1])**.** *Let H be any Hopf algebra (not necessarily finite-dimensional). Then any finite-dimensional sub-bialgebra of H is a Hopf subalgebra of H.*

Before we present examples of Hopf algebras, we introduce the concept of *normal Hopf subalgebra* of a Hopf algebra.

Definition 2.2.16 ([39, Definition 3.4.1]). *Let H be any Hopf algebra. The left adjoint action of H on itself is given by*

$$(ad_l h)(g) = \sum h_1 g S(h_2), \quad \forall h, g \in H.$$

The right adjoint action of H on itself is given by

$$(ad_r h)(g) = \sum S(h_1) g h_2, \quad \forall h, g \in H.$$

A Hopf subalgebra K of H is called normal if both

$$(ad_l H)(K) \subseteq K \, and \, (ad_r H)(K) \subseteq K.$$

Also, for normal Hopf subalgebras we have the following lemma.

Lemma 2.2.17 ([39, Lemma 3.4.2]). *Let H be a Hopf algebra and K a normal Hopf subalgebra of H. Then $HK^+ = K^+H = I$ is a Hopf ideal of H, and $\pi : H \to H/I$ is a homomorphism of Hopf algebras.*

In what follows, we present a few examples of Hopf algebras. The first and most basic example is the group algebra, which is going to be useful in this thesis.

Example 2.2.18 ([17, Example 4.3.1]). *Let G be any group. We already have seen that the group algebra $F[G]$ is a bialgebra (Example 2.2.4). The linear map $S : F[G] \to F[G]$ given by $S(g) = g^{-1}$ defines an antipode for $F[G]$ and then $F[G]$ becomes a Hopf algebra.*

Example 2.2.19 ([17, Example 4.3.1]). *Let G be a finite group. By the previous example, $F[G]$ is a Hopf algebra. Then, by Proposition 2.2.14, $F[G]^*$ is also a Hopf algebra with antipode S^*. We can describe the Hopf algebra structure of $F[G]^*$ by considering the canonical basis G of $F[G]$ and so its corresponding dual basis $\{p_g \mid g \in G\}$ of $F[G]^*$. That is, $p_g(h) = \delta_{g,h}$, for all $g, h \in G$, where $\delta_{g,h}$ is the Kronecker symbol. Then the Hopf algebra structure of*

$H = (F[G])^*$ can be describe as

$$p_g p_h = \delta_{g,l} p_g, \quad \Delta(p_g) = \sum_{h \in G} p_{gh^{-1}} \otimes p_h$$

$$1_{F[G]^*} = \sum_{g \in G} p_g, \quad \epsilon(p_g) = \delta_{1,g}, \quad S(p_g) = p_{g^{-1}}$$

for all $g, h \in G$.

Example 2.2.20. *Let H and K be Hopf algebras. Then $H \otimes K$ has a Hopf algebra structure, where the antipode is given by $S_H \otimes S_K$.*

The next example is a very important one in this thesis and will be revisited later. It is a non-commutative, non-cocommutative Hopf algebra of dimension 8, which will be denoted by H_8. Its construction is due to Kac and Paljutkin, who discovered such Hopf algebra in the 1960's (see [23]).

Example 2.2.21 ([23, 37]). *H_8 is the algebra over F generated by x, y, and z subject to the following relations*

$$\begin{gathered} x^2 = 1, \quad y^2 = 1, \quad xy = yx \\ z^2 = \frac{1}{2}\left(1 + x + y - xy\right), \quad zx = yz, \quad zy = xz. \end{gathered}$$

H_8 has a coalgebra structure with

$$\begin{aligned} \Delta(x) &= x \otimes x, & \epsilon(x) &= 1 \\ \Delta(y) &= y \otimes y, & \epsilon(y) &= 1 \\ \Delta(z) &= \tfrac{1}{2}\left(1 \otimes 1 + x \otimes 1 + 1 \otimes y - x \otimes y\right)(z \otimes z), & \epsilon(z) &= 1. \end{aligned}$$

H_8 becomes a Hopf algebra by setting $S(x) = x$, $S(y) = y$, and $S(z) = z$.

2.2.3 Modules and comodules

In this subsection, we present the concept of modules over an algebra A in terms of commutative diagrams and then, by dualizing such diagrams, we introduce the concept of comodules over a coalgebra C.

We start with the definition of a module over an algebra.

Definition 2.2.22. *Let A be an algebra. A left A-module is a pair (X, γ), where X is a vector space and $\gamma : A \otimes X \to X$ is a homomorphism of vector spaces such that the following diagrams are commutative:*

$$\begin{tikzcd}
A \otimes A \otimes X \arrow[r, "id_A \otimes \gamma"] \arrow[d, "\mu \otimes id_X"'] & A \otimes X \arrow[d, "\gamma"] \\
A \otimes X \arrow[r, "\gamma"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \otimes X \arrow[dd, "\gamma"] \\
F \otimes X \arrow[ur, "\eta \otimes id_X"] \arrow[dr, "\simeq"'] & \\
 & X.
\end{tikzcd}$$

Analogously, one can define a right A-module.

In general, for all $a \in A$ and $m \in X$, we write $a \cdot m$ instead of $\gamma(a \otimes m)$. The first diagram just means that $a \cdot (b \cdot m) = (ab) \cdot m$ and the second diagram can be read as $1_A \cdot m = m$, for all $a, b \in A$ e $m \in X$.

Now we dualize this definition to get the concept of a C-comodule, for C a coalgebra.

Definition 2.2.23. *Let C be a coalgebra. A right C-comodule is a pair (M, ρ), where M is a vector space and $\rho : M \to M \otimes C$ is a homomorphism of vector spaces such that the following diagrams are commutative:*

$$\begin{tikzcd}
M \arrow[r, "\rho"] \arrow[d, "\rho"'] & M \otimes C \arrow[d, "id_M \otimes \Delta"] \\
M \otimes C \arrow[r, "\rho \otimes id_C"'] & M \otimes C \otimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[dr, "\simeq"] \arrow[dd, "\rho"'] & \\
 & M \otimes \\
M \otimes C. \arrow[ur, "id_M \otimes \epsilon"'] &
\end{tikzcd}$$

Analogously, one can define a left C-comodule.

As for coalgebras, we also have a Sweedler notation for comodules. Given $m \in M$, we write $\rho(m) = \sum m_0 \otimes m_1$, where $m_0 \in M$ and $m_1 \in C$.

With this Sweedler notation, the commutativity of the diagrams are saying that

$$\sum m_0 \otimes m_{1_1} \otimes m_{1_2} = \sum m_{0_0} \otimes m_{0_1} \otimes m_1 := \sum m_0 \otimes m_1 \otimes m_2 \tag{2.1}$$

and

$$\sum m_0 \epsilon(m_1) = m. \tag{2.2}$$

For left comodules, the Sweedler notation is given by $\lambda(m) = \sum m_{-1} \otimes m_0$. For more on this, see [17, Chapter 2].

Given a coalgebra C, hereafter we are going to omit the map ρ for a right C-comodule (M, ρ).

Example 2.2.24. *Every coalgebra (C, Δ, ϵ) is a right C-comodule with $\rho = \Delta$.*

Example 2.2.25. *Given a coalgebra C and a vector space X, then $X \otimes C$ is a right C-comodule with $\rho : X \otimes C \to X \otimes C \otimes C$ given by $\rho = id \otimes \Delta$.*

Now, we recall the definition of a module homomorphism in terms of a diagram and then we dualize such notion to obtain the definition of a comodule homomorphism.

Definition 2.2.26. *Let A be an algebra. Let also (X, γ) and (Y, κ) be two left A-modules. A linear map $f : X \to Y$ is a homomoprhism of left A-modules if the following diagram commutes:*

$$\begin{tikzcd} A \otimes X \arrow[r, "id_A \otimes f"] \arrow[d, "\gamma"'] & A \otimes Y \arrow[d, "\kappa"] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

Definition 2.2.27. *Let C be a coalgebra. Let (M, ρ) and (N, ϕ) be two right C-comodules. A linear map $g : M \to N$ is a homomorphism of right C-comodules if the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "\rho"'] & N \arrow[d, "\phi"] \\ M \otimes C \arrow[r, "g \otimes id_C"'] & N \otimes C. \end{tikzcd}$$

In Sweedler notation, we have

$$\phi(g(m)) = \sum g(m_0) \otimes m_1, \text{ for all } m \in M.$$

Although, in this subsection, we are just considering right C-comodules, all the definitions, results etc. can be considered for left C-comodules as well.

We will denote a right (or left) comodule by (M, ρ) and refer to ρ as the right (or left) coaction of C on M.

Definition 2.2.28. *Let M be a right C-comodule. A vector subspace $N \subseteq M$ is called a right C-subcomodule if $\rho(N) \subseteq N \otimes C$.*

Note that, in this case, N is itself a right C-comodule with the coaction given by the restriction of ρ. Moreover, the inclusion map $i : N \hookrightarrow M$ is a C-comodule homomorphism.

Naturally, for a C-subcomodule N of a C-comodule (M, ρ), as one should expect, we have a structure of right C-comodule in the quotient space M/N. This is expressed in the following result.

Proposition 2.2.29 ([17, Proposition 2.1.14])**.** *There exists a unique structure of a right C-comodule on M/N for which the canonical projection $\pi : M \to M/N$ is a homomorphism of right C-comodules.*

In this case, the coaction of C on M/N is given by $\overline{\rho}(\overline{m}) = \sum \overline{m_0} \otimes m_1$, for all $m \in M$, where $\overline{m} = m + N = \pi(m)$. Also, in a more general setting, we have the following proposition.

Proposition 2.2.30 ([17, Proposition 2.1.16])**.** *Let C be a coalgebra and let M and N be two right C-comodules with $f : M \to N$ a comodule homomorphism. Then $\mathrm{Im}(f)$ is a C-subcomodule of N and $\mathrm{Ker}(f)$ is a C-subcomodule of M.*

We saw that for a coalgebra C the dual space C^* has a structure of algebra and also that, given an algebra A, the finite dual A° has a structure of coalgebra. This allows us to establish a sort of duality between coaction and actions.

Proposition 2.2.31 ([39, Lemma 1.6.4])**.**

1) *If M is a right C-comodule, then M is a left C^*-module;*

2) *Let M be a left A-module. Then M is a right A°-comodule if and only if $\{A \cdot m\}$ is finite-dimensional for all $m \in M$, where $A \cdot m$ denotes the action of A on the element m.*

2.2.4 Integrals and semisimplicity

We start this subsection by recalling what a semisimple artinian ring is and then we shall say that a Hopf algebra H is *semisimple* if it is semisimple as an algebra.

Let R be a ring and M be a left (right) R-module. We say that M is a *semisimple module* if M is a direct sum of simple sub-modules, or, equivalently (see [51, 20.2]), every submodule of M is a direct summand of M. A ring R is *semisimple artinian* if it is satisfies any of the following equivalent conditions:

i) R is semisimple as a left R-module;

ii) (**Wedderburn Artin Theorem**) There exist positive integers $t, m_1, \dots, m_t$ such that $R \cong M_{n_1}(D_1) \times \cdots \times M_{n_t}(D_t)$ as rings, where $M_{n_i}(D_i)$ is a matrix ring and D_i is a division ring;

iii) R is semisimple as a right R-module.

For more on this, see [51, chapter 4].

Now, in the Hopf algebra setting, we just say that a Hopf algebra H is semisimple if it is semisimple as an algebra.

We next discuss the relationship between the semisimplicity of a finite-dimensional Hopf algebra and the antipode and *integrals* of the Hopf algebra.

Definition 2.2.32. *Let H be a Hopf algebra. A left integral in H is an element $t \in H$ such that $ht = \epsilon(h)t$, for all $h \in H$; Similarly, a right integral in H is an element $t' \in H$ such that $t'h = \epsilon(h)t'$, for all $h \in H$.*

We denote by $\int_H^l$ the set of left integrals in H and by $\int_H^r$ the set of right integrals in H. Such sets can be easily check to be left and right ideals of H respectively. The next result is due to Larson and Sweedler [33].

Theorem 2.2.33 ([46, Theorem 2.3]). *Let H be a finite-dimensional Hopf algebra. Then*

1) $\dim \int_H^l = 1 = \dim \int_H^r$;

2) The antipode S is bijective, and $S\left(\int_H^l\right) = \int_H^r$.

Example 2.2.34. *Let G be a finite group and $H = F[G]$ its group algebra.*

i) The element $t = \sum\limits_{g \in G} g$ generates the space of left and right integrals in H;

ii) For $H' = F[G]^*$, the element $t = p_1$, as described in Example 2.2.19, generates the space of left and right integrals in H'.

We recall now the definition of separability for an algebra.

Definition 2.2.35. *An algebra A is separable if there exists an element* $e = \sum_{i=1}^{n} a_i \otimes b_i \in A \otimes A$ *such that* $\sum_{i=1}^{n} a_i b_i = 1_A$*, and, for all* $x \in A$, $\sum_{i=1}^{n} x a_i \otimes b_i = \sum_{i=1}^{n} a_i \otimes b_i x$.

We say that a Hopf algebra H is *separable* if it is separable as an algebra.

The next result, due to Larson and Sweedler, deals with integrals and theirs relationship with the semisimplicity of a Hopf algebra.

Theorem 2.2.36 ([46, Theorem 3.2])**.** *Let H be a finite-dimensional Hopf algebra. Then the following statements are equivalent.*

1) H is semisimple;

2) H is separable;

3) $\epsilon(\int_H^l) \neq 0$ (if and only if $\epsilon(\int_H^r) \neq 0$).

This generalizes Maschke's Theorem for group algebras of finite groups, which says that the group algebra $F[G]$ is semisimple if and only if $|G|^{-1} \in F$. In terms of integrals, since $t = \sum_{g \in G} g \in \int_{F[G]}^l$ (and also $t \in \int_{F[G]}^r$), we have $\epsilon(t) = |G|$. Thus, $|G|^{-1} \in F$ if and only if $\epsilon(t) \neq 0$.

Semisimplicity of finite-dimensional Hopf algebras over fields of characteristic zero also can be deduced in terms of the antipode of the Hopf algebra. This is precisely what was proved by Larson and Radford in [32] and it is a characterization of semisimple Hopf algebras.

Theorem 2.2.37 (Larson-Radford)**.** *Let H be a finite-dimensional Hopf algebra over a field F of characteristic zero. Then the following are equivalent:*

i) H is a semisimple Hopf algebra;

ii) H^* is a semisimple Hopf algebra;

iii) $S^2 = id$.

Remark 2.2.38 ([17, Lemma 5.3.1])**.** *In [50, Corollary 2.7], Sweedler proved that if a Hopf algebra contains a non-zero finite-dimensional right ideal, then the Hopf algebra is finite-dimensional. Since, for H a semisimple Hopf algebra, we have that* $H \cong \ker(\epsilon) \oplus I$ *as* *right H-modules. Since* $\ker(\epsilon)$ *has codimension* 1*, it follows that H contains a non-zero finite-dimensional right ideal. Therefore, we conclude that every semisimple Hopf algebra is finite-dimensional.*

As a corollary of Theorem 2.2.37, we will deduce that commutative and cocomutative Hopf algebras are *trivial*, i.e., they are isomorphic to a group algebra or the dual of a group algebra.

Corollary 2.2.39. *Let H be a finite-dimensional Hopf algebra and suppose that F is algebraically closed field of characteristic zero.*

i) If H is commutative, then $H \cong F[G]^*$ for some group G with $\dim(H) = |G|$*;*

ii) If H is cocommutative, then $H \cong F[G]$ for some group G with $\dim(H) = |G|$*.*

Proof. To prove the first item, we note first that H commutative implies that $S^2 = id$. Indeed, for all $h \in H$, we have

$$\begin{aligned}(S * S^2)(h) &= \sum S(h_1)S^2(h_2)\\ &= \sum S^2(h_2)S(h_1)\\ &= \sum S(h_1 S(h_2))\\ &= \epsilon(h)S(1_H) = \epsilon(h)1_H.\end{aligned}$$

Therefore, S^2 is the convolution inverse of the antipode S and so it must be equal to id. Then, by Theorem 2.2.37, H is semisimple. Hence, since H is a commutative semisimple finite-dimensional Hopf algebra over an algebraically closed field, by the Wedderburn-Artin Theorem, H is isomorphic as rings to the direct product of n copies of F, where n is the dimension of H. Thus, H has a complete set of orthogonal idempotents $\{e_1, \cdots, e_n\}$. Let $\{f_1, \cdots, f_n\}$ be the correspondent dual basis in H^*. By Remark 2.1.13, for all $1 \leq i \leq n$, we have

$$\Delta_{H^*}(f_i) = \sum_{l,j=1}^{n} f_i(e_l e_j) f_l \otimes f_j = f_i \otimes f_i.$$

This implies that the basis $\{f_1, \cdots, f_n\}$ of H^* consists of group-like elements. Therefore, H^* is a group algebra, namely, $H^* = F[G(H^*)]$. Then, since $(H^*)^* \cong H$ as Hopf algebras, it follows that $H \cong F[G]^*$ for some group G with $\dim(H) = |G|$.

For the second item, we just have to note that if H is cocomutative then H^* is commutative. Then, by the first item, $H^* \cong F[G]^*$ for some group G with $\dim(H) = |G|$. Therefore, by duality, $H \cong F[G]$ for some group G with $\dim(H) = |G|$. □

Given a coalgebra C, the *coradical* of C is the sum of the simple subcoalgebras of C and is denoted C_0 [42, Definition 3.4.1], where by a simple coalgebra D we just mean that D has only (0) and D as subcoalgebras. C is called *cosemisimple* if $C = C_0$. A Hopf algebra H is called cosemisimple if it is cosemisimple as a coalgebra.

Definition 2.2.40 ([39, Definition 2.4.4])**.** *Let H be a Hopf algebra. An element $T \in H^*$ is a left integral on H if for all $f \in H^*$,*

$$f * T = f(1_H)T.$$

Right integrals on H are defined similarly.

In the finite-dimensional setting, this definition is just saying that an integral *on* H is the same as an integral *in* H^*, since $\epsilon_{H^*}(f) = f(1_H)$, for all $f \in H^*$.

We have the following theorem which characterizes cosemisimple Hopf algebras.

Theorem 2.2.41 ([39, Theorem 2.4.6])**.** *If H is any Hopf algebra, then the following is equivalent:*

a) H is cosemisimple;

b) There exists a left integral T on H satisfying $T(1_H) = 1$.

To end this section, we present a theorem connecting semisimple and cosemisimple Hopf algebras over a field of characteristic zero which will be used in the next chapter.

Theorem 2.2.42 ([42, Corollary 13.2.3])**.** *Let H be a finite-dimensional Hopf algebra over a field F of characteristic zero. Then the following are equivalent:*

a) H and H^* are semisimple;

b) H and H^* are cosemisimple.

2.2.5 The Nichols-Zoeller theorem

Finally, to end this section, we present the Nichols-Zoeller Theorem, which partially answers the question: When Hopf algebras are free over their Hopf subalgebras?

This question is one of the famous conjectures proposed by Irving Kaplansky in 1975, see [25]. If the Hopf algebra H is finite-dimensional, then the Nichols-Zoller theorem answers positively this question, in the sense that it always will be the case for any Hopf subalgebra of H.

Theorem 2.2.43 ([40]). *Let H be a finite-dimensional Hopf algebra, and let $R \subseteq H$ be a Hopf subalgebra of H. Then H is a free R-module.*

This theorem will be useful in the subsequent chapters.

Also, as a consequence, since for any subgroup G' of a finite group G we have that $F[G']$ is a Hopf subalgebra of $F[G]$, the following corollary generalizes the Lagrange Theorem in group theory to the context of Hopf algebras.

Corollary 2.2.44 (Lagrange's Theorem for Hopf algebras). *Let H be a finite-dimensional Hopf algebra, and let $R \subseteq H$ be a Hopf subalgebra of H. Then* $\dim(R)$ *divides* $\dim(H)$.

2.3 Hopf algebra actions on algebras

In this section, we present the concept of *module algebras*, and of *comodule algebras*. Basically, these are concepts of actions of Hopf algebras on algebras and of coactions of Hopf algebras on algebras. Afterwards, we will see that with a Hopf action of an algebra H on an algebra A, it is possible to construct a new algebra called *smash product* and denoted by $A\#H$.

2.3.1 Module algebras and comodule algebras

Since a Hopf algebra H has an algebra structure, we can consider H-modules.

We start with the definition of a *Hopf action*.

Definition 2.3.1. *Let H be a Hopf algebra and A be an algebra. We say that H measures A if there exists a linear map $\cdot : H \otimes A \to A$ satisfying the following conditions*

1) $h \cdot (ab) = \sum_h (h_1 \cdot a)(h_2 \cdot b)$;

2) $h \cdot 1_A = \epsilon(h) 1_A$,

for all $h \in H$, *and* $a, b \in A$, *where* $h \cdot a = \cdot(h \otimes a)$. *Moreover, if* $\cdot : H \otimes A \to A$ *defines an* H*-module structure on* A *and* H *measures* A, *we say that* A *is a left* H*-module algebra, or that there is a Hopf action of* H *on* A.

Remark 2.3.2. *Let* H *be a Hopf algebra acting on an algebra* A. *Let* $0 \neq g \in G(H)$ *be a group-like element of* H. *Then, since* $\Delta(g) = g \otimes g$, *we must have that*

$$g \cdot (ab) = (g \cdot a)(g \cdot b), \text{ for all } a, b \in A.$$

Also, note that $S(g) = g^{-1}$ *is also a group-like element of* H. *Hence,* g *acts as an automorphism of* A, *i.e., there exists* $\alpha_g \in \mathrm{Aut}(A)$, *such that* $g \cdot a = \alpha_g(a)$, *for all* $a \in A$. $\alpha_g(1_A) = 1_A$ *and* $\alpha_g^{-1} = \alpha_{g^{-1}}$.

This concept of an action of a Hopf algebra H on an algebra A generalizes the concept of group actions on algebras.

Example 2.3.3. *We say that a group* G *acts on an algebra* A *by automorphisms if there exists a group homomorphism* $G \to Aut(A)$, *where* $Aut(A)$ *denotes the group of automorphisms of* A. *Let* $H = F[G]$ *be the group algebra of* G, *which is a Hopf algebra as we already have seen. Then an algebra* A *is a left* H*-module algebra if and only if* G *acts on* A *by automorphisms.*

Indeed, if $\phi : G \to Aut(A)$ *is a homomorphism of groups, we define the* H*-module structure on* A *by* $g \cdot a = \phi(g)(a)$, *for all* $g \in G$ *and for all* $a \in A$. *Since* $\Delta(g) = g \otimes g$ *for all* $g \in G$, *and* $\phi(g)$ *is a homomorphism of algebras, it follows that* A *has a left* H*-module algebra structure.*

Reciprocally, suppose that A *has a left* H*-module algebra structure. Define* $\phi : G \to Aut(A)$ *by* $\phi(g)(a) = g \cdot a$, *for all* $g \in G$, $a \in A$. *Since* A *has a left* H*-module structure,* $\phi(gl) = \phi(g)\phi(l)$, *for all* $g, l \in G$. *Also, by Remark 2.3.2, for* $\alpha_g = \phi(g)$, *we get that* $\phi(g)$ *is in fact an automorphism. Moreover,* $\phi(g)(1_A) = g \cdot 1_A = \epsilon(g)1_A = 1_A$. *Therefore,* ϕ *defines an action of* G *on* A *by automorphisms.*

Remark 2.3.4. *Let H be a Hopf algebra and V and W two H-modules. Then, the diagonal action $\cdot : H \otimes V \otimes W$ given by $h \cdot (v \otimes w) = \sum h_1 \cdot v \otimes h_2 \cdot w$, for all $h \in H$, $v \in V$, and $w \in W$, defines a H-module structure on $V \otimes W$.*

Lemma 2.3.5 ([17, Proposition 6.1.4])**.** *Let A be an algebra which is also a left H-module. Then A is an H-module algebra if and only if the multiplication map $\mu : A \otimes A \to A$ is an homomorphism of H-modules.*

Before we introduce the concept of coaction, we present the subspace of invariants of an action.

We say that an action of a Hopf algebra H on an algebra A is *trivial* if $h \cdot a = \epsilon(h)a$, for all $h \in H$, $a \in A$. Let A be an H-module algebra. The subspace $A^H = \{a \in A : h \cdot a = \epsilon(h)a, \ \forall h \in H\}$ is called the *algebra of invariants* and it is a subalgebra of A.

There is also a dual notion for Hopf actions. We briefly present here the definition of a coaction of a Hopf algebra H on an algebra A and a proposition connecting actions and coactions and the dual space of H.

Definition 2.3.6. *Let H be a Hopf algebra and let A be an algebra with a right H-comodule structure (with coaction given by $\rho : A \to A \otimes H$, with $\rho(a) = \sum a_0 \otimes a_1$, for all $a \in A$). We say that A is a right H-comodule algebra (or that there is a Hopf coaction of H on A) if the following conditions hold*

1) $\sum (ab)_0 \otimes (ab)_1 = \sum a_0 b_0 \otimes a_1 b_1$;

2) $\rho(1_A) = 1_A \otimes 1_H$,

for all $h \in H$, and $a, b \in A$.

Analogously, we define a left coaction. Note that the conditions on the definition above are expressing that ρ is a homomorphism of algebras.

Also, as in the case of H-module algebras, we have the subspace of *coinvariants* for a right H-comodule algebra. For A a right H-comodule algebra the subspace defined as $A^{coH} = \{a \in A : \rho(a) = a \otimes 1_H, \ \forall a \in A\}$ is called the *algebra of coinvariants* of A and it is a subalgebra of A.

To end this subsection, we present a proposition which expresses a duality between actions and coaction of a Hopf algebra on an algebra.

Proposition 2.3.7 ([17, Proposition 6.2.4])**.** *Let H be a finite-dimensional Hopf algebra, and A be an algebra. Then A is a right H-comodule algebra if and only if A is a left H^*-module algebra. Moreover, we also have that $A^{H^*} = A^{coH}$.*

2.3.2 Smash products and crossed products

Given a Hopf algebra H acting on an algebra A, it is possible to construct a new algebra called *smash product*, which is, basically, a generalization of the skew group ring for the group action context.

In this subsection, we shall give the definition of smash products and a generalization of it, namely *crossed products.* We also present two basic results about them.

We start, then, with the definition of smash products.

Definition 2.3.8. *Let H be a Hopf algebra and A be a left H-module algebra. The smash product $A\#H$ is defined as $A\#H = A \otimes H$ as vector space. We write $a\#h$ to denote the element $a \otimes h$, for all $a \in A$, $h \in H$. The multiplication is given by*

$$(a\#h)(b\#g) = \sum a(h_1 \cdot b)\#h_2 g,$$

for all $a, b \in A$, and $g, h \in H$. The unity is given by $1_{A\#H} = 1_A\#1_H$.

For any Hopf algebra H and algebra A, the tensor product $A \otimes H$ has a structure of smash product, where the action of H on A is the trivial one, i.e., $h \cdot a = \epsilon(h)a$, for all $h \in H$, $a \in A$.

The next proposition states that $A\#H$ is indeed an algebra and that A and H are subalgebras of the smash product $A\#H$.

Proposition 2.3.9 ([17, Proposition 6.1.7])**.** *Let H be a Hopf algebra and A be a left H-module algebra. Then the following holds:*

i) $A\#H$, with the multiplication defined above, is an algebra;

ii) The maps $a \mapsto a\#1_H$, for all $a \in A$, and $h \mapsto 1_A\#h$, for all $h \in H$, are injective homomorphisms of algebras;

iii) $A\#H$ is free as a left A-module, and if $\{h_i\}_{i \in I}$ is a basis of H, then $\{1_A\#h_i\}_{i \in I}$ is an A-basis of $A\#H$ as a left A-module.

In the sequel, we briefly present the notion of *crossed products* as a generalization of smash products, where the action of the Hopf algebra is twisted by a cocycle.

Definition 2.3.10 ([39, Definition 7.1.1]). *Let H be a Hopf algebra and A be an algebra. Recall Definition 2.3.1 and assume that H measures A. Let $\sigma \in \mathrm{Hom}_F(H \otimes H, A)$ be an invertible map. The crossed product $A\#_\sigma H$ of A with H is the set $A \otimes H$ as a vector space, with multiplication*

$$(a\#h)(b\#k) = \sum a(h_1 \cdot b)\sigma(h_2, k_1)\#h_3k_2,$$

for all $h, k \in H$ and $a, b \in A$. Here $a\#h$ denotes the element $a \otimes h$.

Lemma 2.3.11 ([39, Lemma 7.1.2]). *$A\#_\sigma H$ is an associative algebra with identity element $1_A\#1_H$ if and only if the following conditions are satisfied:*

1) A is a twisted H-module, i.e., $1 \cdot a = a$, for all $a \in A$, and

$$h \cdot (k \cdot a) = \sum \sigma(h_1, k_1)(h_2k_2 \cdot a)\sigma^{-1}(h_3, k_3),$$

for all $h, k \in H$ and $a \in A$.

2) σ is a cocycle, i.e., $\sigma(h, 1) = \sigma(1, h) = \epsilon(h)1$, for all $h \in H$, and

$$\sum [h_1 \cdot \sigma(k_1, m_1)]\sigma(h_2, k_2m_2) = \sum \sigma(h_1, k_1)\sigma(h_2k_2, m),$$

for all $h, k, m \in H$.

In case σ is trivial, that is, $\sigma(h, k) = \epsilon(h)\epsilon(k)1$, for all $h, k \in H$, then the first condition of the Lemma above implies that A is an H-module and the second condition is trivial. Then, A is an H-module algebra. Also, the definition of multiplication in $A\#_\sigma H$ becomes just the definition of multiplication of the smash product $A\#H$, and hence $A\#_\sigma H = A\#H$ [39, Example 7.1.5].

For more on smash products and crossed products, see Chapter 4 and 7 of [39]. Later, in Chapter 3 of this thesis, we shall return to these concepts.

Chapter 3

Semisimple Hopf actions factoring through group actions

Let H be a Hopf algebra acting on an algebra A. Let I be a Hopf ideal of H such that $I \cdot A = 0$, then one says that the action of H on A *factors through* the quotient Hopf algebra H/I. Moreover, if $H/I \cong F[G]$, for some group G, we say that the action of H on A *factors through a group action.*

In [18, Theorem 5.1], Etingof and Walton showed that any semisimple Hopf algebra action on a commutative domain must factor through a group action, for F an algebraically closed field. Later, in [13, Theorem 4.1], Cuadra, Etingof and Walton showed that any action of a semisimple Hopf algebra H, with the ground field F algebraically closed of characteristic zero, on the nth Weyl algebra $A = A_n(F)$ factors through a group action.

In this chapter, we will analyze the main result of the paper [13] to show that any action of a semisimple Hopf algebra H on an enveloping algebra of a finite-dimensional Lie algebra or on an iterated Ore extension of derivation type $A = F[x_1][x_2; d_2][\cdots][x_n; d_n]$ *factors through a group action.* In order to do this, we will consider inner faithful Hopf actions and use a reduction step, which basically consist of passing from algebras in characteristic zero to algebras in positive characteristic by using residue field of the subring R of the ground field F which is generated by all structure constants of H and the action on A.

It is worth to mention that it has already been outlined in [13, p.2] that these methods could be used to establish more general results on semisimple Hopf actions on quantized algebras. In particular it has been announced in [13, p.2] that their methods will apply to actions on module algebras A such that the resulting algebra A_p, when passing to a field of characteristic p, for large p, is PI and their PI-degree is a power of p. Such algebras include universal enveloping algebras of finite-dimensional Lie algebras and algebras of differential operators of smooth irreducible affine varieties.

3.1 Inner faithful Hopf actions

We start this section with the definition of *inner faithful* actions. Let H be a finite-dimensional Hopf algebra over F. A *representation* of H on an algebra A is an algebra homomorphism $\pi : H \to A$. The following definition was given by Banica and Bichon.

Definition 3.1.1 ([7, Definition 2.7]). *Let $\pi : H \to A$ be a representation of a Hopf algebra H on an algebra A. We say that π is inner faithful if* $\mathrm{Ker}(\pi)$ *does not contain any non-zero Hopf ideal.*

Let M be a left H-module, then, if we consider the endomorphism algebra $\mathrm{End}_F(M)$, we have a representation $\pi : H \to \mathrm{End}_F(M)$ due to M be a H-module. In this case, to say that π is inner faithful is to say that $I \cdot M \neq 0$ for any non-zero Hopf ideal I of H. This leads to the definition below.

Definition 3.1.2 ([11, Definition 1.2]). *Let M be a left H-module. We say that M is an inner faithful H-module (or H acts inner faithfully on M) if $I \cdot M \neq 0$ for any non-zero Hopf ideal I of H. Given a Hopf action of H on an algebra A (i.e., A is a left H-module algebra), we say that this action is inner faithful if the left H-module A is inner faithful.*

For a ring R and a left R-module M, we denote $\mathrm{Ann}_R(M) = \{r \in R \mid rm = 0, \forall m \in M\}$. For a Hopf algebra H acting on an algebra A, $\mathrm{Ann}_H(A) = \{h \in H \mid h \cdot a = 0, \forall a \in A\}$. Then H acts inner faithfully on A if and only if $\mathrm{Ann}_H(A)$ does not contain any non-zero Hopf ideal.

Remark 3.1.3. *The first thing to note is that any Hopf action factors through an inner faithful action. For if H is a Hopf algebra and A is a left H-module algebra, then we can consider the Hopf ideal $I = \sum_{J \subseteq \mathrm{Ann}_H(A)} J$, which is the largest Hopf ideal of H such that $I \cdot A = 0$. Then H/I acts inner faithfully on A.*

Indeed, let $B \subseteq H/I$ be a Hopf ideal such that $B \cdot A = 0$. By Lemma 2.2.12 $B = J'/I$ for some Hopf ideal J' of H with $I \subseteq J'$. Since $B \cdot A = 0$, we must have that $J' \cdot A = 0$, i.e., $J' \subseteq \mathrm{Ann}_H(A)$. Hence, $J' \subseteq I$ and thus $B = 0$.

In what follows, we will collect some results about inner faithful actions.

Proposition 3.1.4. *Let G be any finite group, A be an algebra, and $\psi : G \to Aut(A)$ be a group homomorphism. Then ψ is injective if and only if $F[G]$ acts inner faithfully on A.*

Before we prove this Proposition, we will review here some results which can be found in [41, Lemma 1.3, Lemma 1.8]. We start with a definition.

Definition 3.1.5. *Let N be a subgroup of a group G. A left transversal of N in G is a complete subset of left coset representatives for N in G, i.e., $G = \bigcup_{y \in Y} yN$ as disjoint union. A right transversal is defined similarly.*

Let G be a group and N be a subgroup of G, to prove the following lemma, we define $\pi_N : F[G] \to F[N]$ the natural projection given by

$$\pi_N \left(\sum_{g \in G} \alpha_g g \right) = \sum_{g \in N} \alpha_g g.$$

π_N is an F-linear map and satisfies $\pi_N(ba) = b\pi_N(a)$ and $\pi_N(ab) = \pi_N(a)b$, for all $a \in F[G]$ and $b \in F[N]$ by [41, Lemma 1.2].

Lemma 3.1.6 ([41, Lemma 1.3])**.** *Let N be a subgroup of a group G and Y be a left transversal for N in G. Then every element $\alpha \in F[G]$ can be written uniquely as a finite sum of the form*

$$\alpha = \sum_{y \in Y} y\alpha_y$$

with $\alpha_y \in F[N]$.

Proof. Let $\alpha \in F[G]$ be a non-zero element and write $\alpha = \sum_{g \in G} \alpha_g g$. Let $X = \{g \in G : \alpha_g \neq 0\}$. Since $|X| < \infty$, there exists a finite number of left cosets of N, say $y_1N, \dots, y_nN$, with $y_i \in Y$, whose union contains X. Then, we can write $\alpha = \sum_{i=1}^{n} \alpha_i$, where α_i is the partial sum of those $\alpha_g g$ with $g \in y_iN$. But $g \in y_iN$ implies that $y_i^{-1} g \in N$. So $y_i^{-1}\alpha_i \in F[N]$ for all $i \in \{1, \dots, n\}$. Therefore, since $\alpha = \sum_{i=1}^{n} y_i(y_i^{-1}\alpha_i)$, $\alpha = \sum_{y \in Y} y\alpha_y$, with $\alpha_y \in F[N]$.

To prove the uniqueness, if $\sum_{y\in Y} ya_y = \sum_{y\in Y} yb_y$, with $a_y, b_y \in F[N]$, are two finite sums, then, for every $y_0 \in Y$ and $y \in Y$, since $y_0^{-1}y \in N$ if and only if $y = y_0$, we must have

$$\pi_N\left(\sum_{y\in Y} y_0^{-1}ya_y\right) = y_0^{-1}y_0a_{y_0} = a_{y_0} \quad \text{and} \quad \pi_N\left(\sum_{y\in Y} y_0^{-1}yb_y\right) = y_0^{-1}y_0b_{y_0} = b_{y_0}.$$

Therefore, since $\pi_N\left(\sum_{y\in Y} y_0^{-1}ya_y\right) = \pi_N\left(\sum_{y\in Y} y_0^{-1}yb_y\right)$, the uniqueness is proved. □

Note that if Y is a right transversal, then we have a similar Lemma. But instead of writing $\alpha = \sum_{y\in Y} y\alpha_y$ with $\alpha_y \in F[N]$, we write $\alpha = \sum_{y\in Y} \alpha_y y$ with $\alpha_y \in F[N]$.

Before we continue, we note a Hopf algebra fact for group algebras. Let G be a group and N a normal subgroup of G. Then, as we noticed before, $F[N]$ is a Hopf subalgebra of $F[G]$. Furthermore, $F[N]$ is a normal Hopf subalgebra (Definition 2.2.16) of $F[G]$. Indeed, since N is a normal subgroup of G, then $gng^{-1} \in N$ for all $n \in N$, and $g \in G$. This implies that $(ad_l F[G])(F[N]) \subseteq F[N]$ and $(ad_r F[G])(F[N]) \subseteq F[N]$. The next lemma is due to Rolf Farnsteiner.

Lemma 3.1.7. *Let $H = F[G]$ be the group algebra of a finite group G and $I \subsetneq H$ be a Hopf ideal. Then there exists a normal subgroup $N \lhd G$ such that $I = (F[G])(F[N])^+$.*

Proof. Let $I \subsetneq H$ be a Hopf ideal and consider the projection map $\pi : H \to H/I$, which is a homomorphism of Hopf algebras. H/I is generated, as vector space, by $\pi(G(H)) \subseteq G(H/I)$, since $G(H) = G$, $\text{span}(\pi(G)) = H/I$. Since $\text{span}(\pi(G)) \subseteq \text{span}(G(H/I)) \subseteq H/I = \text{span}(\pi(G))$, and since the distinct grouplike elements of a Hopf algebra are linearly independent (Lemma 2.1.7), we conclude that $\pi(G) = G(H/I)$ and $\dim(H/I) = |G(H/I)|$.

Consider $N := \text{Ker}\,(\pi|_G)$, which is a normal subgroup of G. Then, $|\pi(G)| = |G/N|$ and therefore $\dim(H/I) = |G/N|$. The normality of N in G implies that $F[N]$ is a normal Hopf subalgebra of $F[G]$. Then, $J_N := (F[G])(F[N])^+$ is a Hopf ideal of H and $J_N \subset I$. Thus, π induces a surjection $\psi : F[G]/J_N \to F[G]/I$ of Hopf algebras.

Claim: $F[G]/J_N \cong F[G/N]$

In fact, let Y be a right transversal for N in G. By Lemma 3.1.6, for $\alpha \in F[G]$, we can write $\alpha = \sum_{y\in Y} \alpha_y y$ with $\alpha_y \in F[N]$. Now, consider the natural projection $p : G \to G/N$, which is a group homomorphism, and then the induced Hopf algebras homomorphism $p_N : F[G] \to$

$F[G/N]$. So $p_N(\alpha) = \sum_{y \in Y} p_N(\alpha_y) p_N(y)$. Since $\alpha_y \in F[N]$, $p_N(\alpha_y) \in F\bar{1}$ and since Y is a right transversal for N in G, $\{p_N(y)\}_{y \in Y}$ is a basis for $F[G/N]$. Thus,

$$p_N(\alpha) = 0 \Leftrightarrow p_N(\alpha_y) = 0 \ \forall y \in Y \Leftrightarrow \epsilon(\alpha_y) = 0 \ \forall y \in Y \Leftrightarrow \alpha_y \in F[N]^+, \forall y \in Y.$$

Hence, $\alpha \in \operatorname{Ker} p_N \Leftrightarrow \alpha \in (F[N])^+ F[G] = F[G](F[N])^+$, i.e., $\operatorname{Ker} p_N = J_N$. Also, p_N is clearly surjective. Therefore, $(F[G])/J_N \cong F[G/N]$ and the claim is proved.

Thus, $\dim((F[G])/J_N) = \dim((F[G])/I) = |G/N|$. Hence, $J_N = I$ as claimed.

□

Now we are able to prove Proposition 3.1.4.

Proof of Proposition 3.1.4. $(\Leftarrow)$

Let $N = \operatorname{Ker}(\psi)$ be the kernel of ψ. Since N is a normal subgroup of G, then $R = F[N]$ is a normal Hopf subalgebra of H. Therefore, $J = HR^+$ is a Hopf ideal of H.

Note that for all $x \in R$, since N is the kernel of ψ, the action of x on A is trivial, i.e., $x \cdot a = \epsilon(x)a \ \forall a \in A$. So $J \cdot A = \{0\}$ and thus, since the action is inner faithful, $J = \{0\}$, which implies $R^+ = \{0\}$. Let $y \in N$. Then $y - e \in R^+$. Hence, $N = \{e\}$, i.e., ψ is injective.

$(\Rightarrow)$

Let $I \lhd F[G]$ be a Hopf ideal such that $I \cdot A = 0$. By Lemma 3.1.7, $I = (F[G])(F[N])^+$ for some normal subgroup $N \lhd G$. Let $x \in N$, thus $x - e \in (F[N])^+$ and so $x - e \in I$. Then, $(x - e) \cdot a = 0$ for all $a \in A$, i.e., $x \cdot a = a$ for all $a \in A$. This implies $\psi(x) = \psi(e)$ and thus $x = e$ as ψ is injective. Therefore, $N = \{e\}$ and so $(F[N])^+ = 0$. Hence, $I = 0$. Thus, $F[G]$ acts inner faithfully on A. □

Let H be a Hopf algebra acting on an algebra A. Let also $I \subseteq \operatorname{Ann}_H(A)$ be a Hopf ideal of H such that H/I is a group algebra. Since $I \subseteq \operatorname{Ann}_H(A)$, we have an action of $H/I \cong F[G]$, for some group G, on A. Consider the induced group homomorphism $\psi : G \to Aut(A)$. As a consequence of Lemma 3.1.7, we have the following corollary.

Corollary 3.1.8. *Let I be a Hopf ideal contained in $\operatorname{Ann}_H(A)$ such that $H/I \cong F[G]$ is a group algebra and denote by ψ the induced homomorphism $\psi : G \to Aut(A)$. Then ψ is injective if and only if I is the largest Hopf ideal contained in $\operatorname{Ann}_H(A)$.*

Proof. Assume that I is the largest Hopf ideal contained in $\mathrm{Ann}_H(A)$. Let B be a Hopf ideal of H/I with $B \cdot A = 0$. By Lemma 2.2.12, there exists a Hopf ideal J of H containing I such that $B = J/I$. But $(J/I) \cdot A = 0$ implies that $J \cdot A = 0$. So $J = I$. Therefore, the action of H/I is inner faithful and thus, by Proposition 3.1.4, ψ is injective.

Conversely, if ψ is injective, then by Proposition 3.1.4 $H/I \cong F[G]$ acts inner faithfully on A. Now, consider the Hopf ideal $J' = \sum_{I' \subseteq \mathrm{Ann}_H(A)} I'$, for I' Hopf ideals. Then, $I \subseteq J'$ and J'/I is a Hopf ideal of H/I such that $J'/I \cdot A = 0$. Since the action is inner faithful, it follows that $J'/I = 0$, i.e., $J' = I$ and hence the Corollary is proved. $\square$

Remark 3.1.9. *If $A \subseteq B$ is an extension of left H-modules, for H a Hopf algebra, and H acts inner faithfully on A, then H also acts inner faithfully on B, because* $\mathrm{Ann}_H(B) \subseteq \mathrm{Ann}_H(A)$.

Inner faithful actions can be extended to rings of quotients of a ring. For instance, in [13, Lemma 3.1], the authors showed that if a PI domain B admits an inner faithful action of a Hopf algebra H, then so does its quotient division algebra Q_B.

We are going to prove a similar result for the Martindale ring of quotients. To do so, we recall some results and constructions about Hopf algebra actions and Martindale quotient rings.

Let R be a ring and let $\mathcal{F}$ be the filter of ideals of R which have zero annihilator. Let $S = \bigcup_{I \in \mathcal{F}} \mathrm{Hom}({}_R I, {}_R R)$ with the equivalence relation define as $f \sim g$ if and only if $f = g$ on some $K \in \mathcal{F}$, $K \subseteq \mathrm{Dom}(f) \cap \mathrm{Dom}(g)$. The *left Martindale* ring of quotients of R is the quotient set $Q^l(R) = S/\sim$. Denoting the equivalent classes of $Q^l(R)$ by (I, f), where $f : I \to R$, then for (I, f) and (J, g), $(I \cap J, f + g)$ defines the addition and $(IJ, f \circ g)$ the multiplication and so $Q^l(R)$ becomes a ring (see [31, Proposition 14.9]). Also, R embeds into $Q^l(R)$ as right multiplications on $I = R$. Using right R-module maps, the *right Martindale* quotient ring $Q^r(R)$ is defined analogously.

For $R = A$ an H-module algebra, we consider $\mathcal{F}_H$, the filter of H-stable ideals of A with zero annihilator instead of $\mathcal{F}$. With the same constructions as before, we obtain a ring denoted by $Q^l_H(A)$. And the same is true for the right Martindale quotient ring, denoted by $Q^r_H(A)$.

Suppose that H has a bijective antipode, in [12], Cohen showed that $Q^l_H(A)$ and $Q^r_H(A)$ are H-module algebras, i.e., the action of H on A can be extended to the Martindale ring of quotients $Q^l_H(A)$ and $Q^r_H(A)$. For (I, f) an element of $Q^l_H(A)$, and $h \in H$, the action

$h \cdot f : I \to A$ is defined by

$$(a)(h \cdot f) = \sum h_2 \cdot [(S^{-1}(h_1) \cdot a)f] \quad \forall a \in I,$$

where we write the arguments on the left side. For $g : I \to A$, $I \in \mathcal{F}_H$, an element of $Q^r_H(A)$, and $h \in H$, the action $h \cdot g : I \to A$ is defined by

$$(h \cdot g)(a) = \sum h_1 \cdot [g(S(h_2) \cdot a)] \quad \forall a \in I.$$

Proposition 3.1.10. *Let H be a Hopf algebra acting on an algebra A. Then, H acts inner faithfully on A if and only if H acts inner faithfully on $Q^l_H(A)$.*

Proof. ($\Rightarrow$) Since $A \subseteq Q^l_H(A)$ is an extension of H-modules, it follows from Remark 3.1.9 that H acts inner faithfully on $Q^l_H(A)$.

($\Leftarrow$) Suppose the action of H on A is not inner faithful. Then, there exists $0 \neq I \subset \mathrm{Ann}_H(A)$ a Hopf ideal. Let $h \in I$ and write $\Delta(h) = \sum h'_i \otimes h_i + g_i \otimes g'_i$, where $h'_i, g'_i \in I$. Then, for any $(J, f) \in Q^l_H(A)$ and $a \in J$

$$(a)(h \cdot f) = \sum h_i \cdot [(S^{-1}(h'_i) \cdot a)f] + g'_i \cdot [(S^{-1}(g_i) \cdot a)f] = 0.$$

So $h \cdot f = 0$ on J, i.e., $I \cdot Q^l_H(A) = 0$. Hence H does not act inner faithfully on $Q^l_H(A)$. Therefore, H acting inner faithfully on $Q^l_H(A)$ implies that H acts inner faithfully on A.

□

A similar result for $Q^r_H(A)$ is proved analogously.

To end this section, we present a Lemma which is essentially contained in the proof of [13, Proposition 2.4].

Lemma 3.1.11 (Cuadra-Etingof-Walton)**.** *A finite-dimensional Hopf algebra H acts inner faithfully on an algebra A if and only if $A^{\otimes n}$ is a faithful left H-module for some $n > 0$.*

Proof. The statement follows from the proof of [13, Proposition 2.4]. Let $\mathrm{Ann}_H(M)$ denote the annihilator of a left H-module M. For any $m > 0$, we set $K_m = \mathrm{Ann}_H(A^{\otimes m})$, where $A^{\otimes m}$ is a left H-module via the diagonal H-action, i.e., $h \cdot (a_1 \otimes \cdots \otimes a_m) = \sum h_1 \cdot a_1 \otimes \cdots \otimes h_m \cdot a_m$, for all $h \in H$ and $a_1, \ldots, a_m \in A$.

For $m \leq n$, $A^{\otimes m}$ embeds into $A^{\otimes n}$ as left H-module via $a_1 \otimes \cdots \otimes a_m \mapsto a_1 \otimes \cdots \otimes a_m \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-m \text{ times}}$, for all $a_1, \ldots, a_m \in A$. Hence, we can conclude that $K_m \supseteq K_n$. Clearly, since H is finite-dimensional, the descending chain of ideals K_m stabilises at some index n. Thus, $K_m = K_n = K_{2n} := K$ for all $m \geq n$.

Considering the componentwise action of $H \otimes H$ on $A^{\otimes n} \otimes A^{\otimes n}$, by [17, Lemma 1.4.8], the annihilator of $A^{\otimes n} \otimes A^{\otimes n}$ is equal to $H \otimes K + K \otimes H$. Since

$$0 = K \cdot A^{\otimes 2n} = \Delta(K) \cdot (A^{\otimes n} \otimes A^{\otimes n}),$$

we get $\Delta(K) \subseteq H \otimes K + K \otimes H$. As $\epsilon(K)1_A = K \cdot 1_A = 0$, we conclude that K is a coideal. Thus K is a bi-ideal and $(H/K)^*$ is a sub-bialgebra of the finite-dimensional Hopf algebra H^*. By Proposition 2.2.15, $(H/K)^*$ is a Hopf subalgebra and hence K is a Hopf ideal of H. As H acts inner faithfully on A, $K = 0$, i.e., $A^{\otimes n}$ is a faithful H-module.

For the converse, we just have to note that if I is a Hopf ideal that annihilates A, then it would also annihilate $A^{\otimes n}$. Hence $I = 0$. □

3.2 Reduction process to positive characteristic

In order to prove that actions of semisimple Hopf algebras on Weyl algebras over an algebraically closed field of characteristic zero factor through a group action, in [13], Cuadra, Etingof and Walton used a *reduction process to positive characteristic.* The goal of this section is to present this reduction process and, in addition, to give an alternative proof for the reduction step in Proposition 3.2.9.

3.2.1 Ring of structure constants of an action

We start this subsection by recalling that an ideal p of a ring R is called *completely prime* if R/p is a domain ([30, p. 206]). Let H be a semisimple Hopf algebra over a field F of characteristic zero. By Theorem 2.2.42, H is also cosemisimple. Suppose that H acts on a domain A which is finitely presented as an F-algebra, i.e.,

$$A \simeq F\langle x_1, \ldots, x_n \rangle / P,$$

with P a finitely generated completely prime ideal of the free algebra $F\langle x_1, \dots, x_n\rangle$. The action of H on A is given by an F-linear map

$$H \longrightarrow \mathrm{End}_F(A), \qquad h \mapsto [a \mapsto h \cdot a].$$

We want to consider only the "essential" ingredients of the interplay between H and A, or, in other words, we will look at the "skeleton" of the situation.

The structure constants of H are the constants that define the Hopf algebra structure of H. First of all, H is finite-dimensional, say of dimension d and hence has an F-basis $\{b_1, \dots, b_d\}$. We may assume that 1_H is one of the basis vectors. The Hopf algebra structure of H is determined by a set of constants $\mu_k^{i,j}, \eta_{i,j}^k, \nu_j^i$ and $\epsilon(b_i)$ such that for all $1 \leq i, j, k \leq d$:

$$b_i \cdot b_j = \sum_{k=1}^{d} \mu_k^{i,j}\, b_k, \qquad \Delta(b_k) = \sum_{i,j=1}^{d} \eta_{i,j}^k\, b_i \otimes b_j, \qquad S(b_i) = \sum_{j=1}^{d} \nu_j^i\, b_j.$$

As H is semisimple, by Theorem 2.2.36, there exists a left integral $t = \sum_{i=1}^{d} \tau_i b_i$ in H with $\epsilon(t) = 1$, and as H is cosemisimple, by Theorem 2.2.41, there exists a left integral $t^* = \sum_{i=1}^{d} \tau_i^* b_i^*$ in H^* with $t^*(1) = 1$ for some $\tau_i, \tau_i^* \in F$, where $\{b_1^*, \dots, b_d^*\}$ denotes the dual basis of H. Moreover, the action of H on A is determined by the images of the action of the basis elements b_i on the algebra generators $\overline{x_j} = x_j + P$ of A, i.e.,

$$b_i \cdot \overline{x_j} = f_{ij}(\overline{x_1}, \dots, \overline{x_n}),$$

where f_{ij} are non-commutative polynomials in $F\langle x_1, \dots, x_n\rangle$. Since P is finitely generated, there are non-commutative polynomials $p_1, \dots, p_m$ such that $P = \langle p_1, \dots, p_m\rangle$.

We consider now the subring R of F generated as $\mathbb{Z}$-algebra by all constants

$$\mu_k^{i,j},\ \eta_{i,j}^k,\ \nu_j^i,\ \epsilon(b_i),\ \tau_i,\ \tau_i^*,\ \text{coefficients of } f_{ij},\ \text{coefficients of } p_1, \dots, p_m. \qquad (\star)$$

We call R the *ring of structure constants of the action of H on A.*

As it was mentioned at the beginning of the previous chapter, Hopf algebras can be constructed over rings. For more on this, see [9, Chapter 2].

Let $H_R = \bigoplus_{i=1}^{d} Rb_i$. Then H_R is a R-Hopf algebra with structure constants $(\star)$. Let $A_R = R\langle x_1, \ldots, x_n\rangle/P'$, where P' is the ideal of $R\langle x_1, \ldots, x_n\rangle$ generated by $p_1, \ldots, p_m$. The action of H on A yields now an action of H_R on A_R, i.e., we have a ring homomorphism $H_R \to \mathrm{End}_R(A_R)$. Since $t \in H_R$ still satisfies $\epsilon(t) = 1$, H_R is separable over R. Analogously as $t^* \in H_R^*$ satisfies $t^*(1) = 1$, H_R^* is separable over R (see [24] or [34, Section 3]).

Note also that $H \simeq H_R \otimes_R F$ and $A \simeq A_R \otimes_R F$ as algebras via $h \otimes_R \alpha \mapsto \alpha h$ and $a \otimes_R \beta \mapsto \beta a$, for all $h \in H_R$, $a \in A_R$, and $\alpha, \beta \in F$. Hence A_R is again a domain. Moreover, A_R is finitely presented.

Before we continue, we recall the definition of *Hilbert rings* (see [20, 29] or [45, Chapter 11]) and some results which will be useful to show that R is a Hilbert ring.

Definition 3.2.1 ([45, 11.67])**.** *A commutative ring R is a Hilbert ring (or Jacobson ring) if every prime ideal in R is an intersection of maximal ideals.*

The next three theorems will be used in the sequel and their proofs can be found in [20] or [45, Chapter 11].

Theorem 3.2.2 ([20, Theorem 2])**.** *If R is a Hilbert ring, and I an ideal of R, then R/I is also a Hilbert ring.*

Theorem 3.2.3 ([45, Theorem 11.69])**.** *A commutative ring R is a Hilbert ring if and only if $R[x]$ is a Hilbert ring.*

Theorem 3.2.4 ([20, Theorem 5])**.** *A ring R is a Hilbert ring if and only if every maximal ideal in $R[x]$ contracts in R to a maximal ideal.*

Since F is a field of characteristic 0, R is an integral domain that contains the integers $\mathbb{Z}$. Moreover, we have a finite number of generators for R as a ring and therefore we can consider R as finitely generated $\mathbb{Z}$-algebra. Let $a_1, \ldots, a_s$ be a set of generators of R as $\mathbb{Z}$-algebra and consider the surjective ring homomorphism

$$\varphi : \mathbb{Z}[y_1, \ldots, y_s] \to R, \qquad y_i \mapsto a_i.$$

Therefore, $R \cong \mathbb{Z}[y_1, \ldots, y_s]/\operatorname{Ker}(\varphi)$ as rings. Since $\mathbb{Z}$ is a Hilbert ring (every prime ideal in $\mathbb{Z}$ is maximal), then, by Theorem 3.2.2 and Theorem 3.2.3, R is a Hilbert ring as well. In particular, the prime ideal 0 of R is equal to $\mathrm{Jac}(R)$, the intersection of maximal ideals of R.

If $\mathfrak{m}$ is any maximal ideal of R, then $\pi \circ \varphi : \mathbb{Z}[y_1, \ldots, y_s] \to R/\mathfrak{m}$, where π is the canonical projection, is a surjective ring homomorphism. Hence, from the first isomorphism theorem for rings, $\mathbb{Z}[y_1, \ldots, y_s]/\operatorname{Ker}(\pi \circ \varphi) \cong R/\mathfrak{m}$. Therefore, $\mathbb{Z}[y_1, \ldots, y_s]/\operatorname{Ker}(\pi \circ \varphi)$ is a field and $\operatorname{Ker}(\pi \circ \varphi) = \varphi^{-1}(\mathfrak{m})$ is a maximal ideal of $\mathbb{Z}[y_1, \ldots, y_s]$. Hence, by Theorem 3.2.4, $\varphi^{-1}(\mathfrak{m}) \cap \mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$, i.e., there exists a prime number p such that $\varphi^{-1}(\mathfrak{m}) \cap \mathbb{Z} = p\mathbb{Z}$. In particular, $R/\mathfrak{m}$ has positive characteristic p. By the Weak Nullstellensatz ([44, §4.10]), $R/\mathfrak{m}$ is a finite field extension of the prime subfield $\mathbb{Z}/p\mathbb{Z}$, i.e. $R/\mathfrak{m}$ is a finite field.

3.2.2 Reduction to positive characteristic

In this subsection, we shall give a new proof of [13, Lemma 2.3], which will be used in the next section.

In order to prove the main proposition, Proposition 3.2.9, we will need some lemmas first.

Lemma 3.2.5. *Let R be an integral domain of characteristic 0 with $\operatorname{Jac}(R) = 0$ such that $\operatorname{char}(R/\mathfrak{m}) > 0$ for all maximal ideals $\mathfrak{m}$ of R. For any $0 \neq a \in R$ and integer $q > 1$, set $X_{a,q} = \{\mathfrak{m} \in \operatorname{MaxSpec}(R) \mid \operatorname{char}(R/\mathfrak{m}) > q \text{ and } a \notin \mathfrak{m}\}$. Then $\bigcap_{\mathfrak{m} \in X_{a,q}} \mathfrak{m} = 0$.*

Proof. Let $0 \neq a \in R$ and $q > 1$ an integer. Set $X = \operatorname{MaxSpec}(R)$, the set of all maximal ideals of R. For each $\mathfrak{m} \in X$, let $p_{\mathfrak{m}} = \operatorname{char}(R/\mathfrak{m})$. Set $B = \{\mathfrak{m} \in X \mid a \notin \mathfrak{m} \text{ and } p_{\mathfrak{m}} \leq q\}$. Now, if $B = \emptyset$, then we can conclude that $a \in \bigcap_{\mathfrak{m} \in X \setminus X_{a,q}} \mathfrak{m}$ and so, since by hypothesis $a \neq 0$ and

$$0 = \operatorname{Jac}(R) = \left(\bigcap_{\mathfrak{m} \in X \setminus X_{a,q}} \mathfrak{m} \right) \cap \left(\bigcap_{\mathfrak{m} \in X_{a,q}} \mathfrak{m} \right),$$

R being a domain implies that $\bigcap_{\mathfrak{m} \in X_{a,q}} \mathfrak{m} = 0$ as we want. If $B \neq \emptyset$, then for any $\mathfrak{m} \in B$, we have $1 < p_{\mathfrak{m}} \leq q$. Therefore, $p_m \in \{p_{\mathfrak{m}_1}, \ldots, p_{\mathfrak{m}_n}\}$, for some $\mathfrak{m}_1, \ldots, \mathfrak{m}_n \in B$. We can define the non-zero element

$$b = \prod_{i=1}^{n} p_{\mathfrak{m}_i}$$

and conclude that $b \in \bigcap_{\mathfrak{m} \in B} \mathfrak{m}$. So, $ab \in \bigcap_{\mathfrak{m} \in X \setminus X_{a,q}} \mathfrak{m}$, because if $a \notin \mathfrak{m} \in X \setminus X_{a,q}$, then $p_{\mathfrak{m}} \leq q$. Hence $p_{\mathfrak{m}}$ divides b and, as $p_{\mathfrak{m}} \in \mathfrak{m}$, we have $ab \in \mathfrak{m}$. By hypothesis, $a \neq 0$ and R being a domain implies $\bigcap_{\mathfrak{m}\in X\setminus X_{a,q}} \mathfrak{m} \neq 0$. However, $0 = \mathrm{Jac}(R) = \left(\bigcap_{\mathfrak{m}\in X\setminus X_{a,q}} \mathfrak{m}\right) \cap \left(\bigcap_{\mathfrak{m}\in X_{a,q}} \mathfrak{m}\right)$ and R being a domain shows $\bigcap_{\mathfrak{m}\in X_{a,q}} \mathfrak{m} = 0$. □

The next two lemmas are linear algebra lemmas. The first one will be used to prove the second one, which in turn will be useful in the proof of the main proposition of this subsection.

Lemma 3.2.6. *Let V be a vector space and $h_1, \dots, h_d$ be linearly independent endomorphisms of V. Then, there exists a finite-dimensional subspace $U_d \subseteq V$ such that the restrictions of $h_1, \dots, h_d$ to U_d are linearly independent.*

Proof. Indeed, by induction on d we show this statement. For $d = 1$, since h_1 must be non-zero, there exists $v_1 \in V$ such that $h_1(v_1) \neq 0$. We define $U_1 = span\{v_1\}$.

Now we assume that the claim is true for $d \geq 1$ and we show that it is also true for $d+1$. By induction hypothesis, we have that there exists a finite-dimensional subspace U_d of V such that the restrictions of $h_1, \dots, h_d$ to U_d are linearly independent. We consider the restriction of h_{d+1} to U_d. If the restrictions of $h_1, \dots, h_d, h_{d+1}$ to U_d are linearly independent, then we set $U_{d+1} = U_d$ and we are done. Otherwise, if the restrictions of $h_1, \dots, h_d, h_{d+1}$ to U_d are linearly dependent, then we should have

$$h_{d+1} = \alpha_1 h_1 + \cdots + \alpha_d h_d \text{ on } U_d, \text{ for some } (\alpha_1, \dots, \alpha_d) \in F^d \tag{3.1}$$

Since the restriction of $h_1, \dots, h_d$ to U_d are linearly independent, we have that the representation given in (3.1) is unique. Now let $N = \mathrm{Ker}(h_{d+1} - \alpha_1 h_1 - \cdots - \alpha_n h_n)$ on V. Since $h_1, \dots, h_{d+1}$, by hypothesis, are linearly independent on V, we have that $N \neq V$. Therefore, there exists $v_{d+1} \in V$ such that $v_{d+1} \notin N$. Now we define U_{d+1} to be the span of U_d and v_{d+1}, which is finite-dimensional. We claim that the restrictions of $h_1, \dots, h_{d+1}$ to U_{d+1} are linearly independent. In fact, if $\beta_1 h_1 + \cdots + \beta_{d+1} h_{d+1} = 0$ on U_{d+1}, then, on U_d, using (3.1), we have that

$$\beta_1 h_1 + \cdots + \beta_d h_d + \beta_{d+1}(\alpha_1 h_1 + \cdots + \alpha_d h_d) = 0 \Leftrightarrow (\beta_1 + \beta_{d+1}\alpha_1) h_1 + \cdots + (\beta_d + \beta_{d+1}\alpha_d) h_d = 0$$

and that implies, since $h_1, \dots, h_d$ are linearly independent on U_d, that $\beta_i = -\beta_{d+1}\alpha_i$ for all $i \in \{1, \dots, d\}$. So, we get that $\beta_{d+1}(h_{d+1} - \alpha_1 h_1 - \alpha_2 h_2 - \cdots - \alpha_d h_d) = 0$ on U_{d+1}, which implies that $\beta_{d+1} = 0$ (because $v_{d+1} \notin N$). Therefore, we have $\beta_1 h_1 + \cdots + \beta_d h_d = 0$ on U_{d+1}. In particular, $\beta_1 h_1 + \cdots + \beta_d h_d = 0$ on U_d, which implies that $\beta_i = 0$ for all $i \in \{1, \ldots, d\}$. Therefore, we get that the restrictions of $h_1, \ldots, h_{d+1}$ to U_{d+1} are linearly independent. $\square$

Finally, the following lemma characterizes when endomorphisms $h_1, \ldots, h_n$ of a vector space V are linearly independent.

Lemma 3.2.7. *Let V be a vector space and let $h_1, \ldots, h_n$ be endomorphisms of V. Then $h_1, \ldots, h_n$ are linearly independent if and only if there exist $v_1, \ldots, v_m \in V$ and $f_{jk} \in V^*$ for $1 \leq j \leq n$ and $1 \leq k \leq m$ such that the matrix*

$$\left(\sum_{k=1}^{m} f_{jk}(h_i(v_k)) \right)_{1 \leq i,j \leq n}$$

has non-zero determinant.

Proof. Let V be a vector space and let $h_1, \ldots, h_n$ be linearly independent endomorphisms of V. Then, by Lemma 3.2.6, there exists a finite-dimensional subspace U of V of dimension $q \leq n$ such that $h_1, \ldots, h_n$ restricted to U are linearly independent. Let $\{v_1, \ldots, v_q\}$ be a basis for U. For each $1 \leq i \leq n$ we define $\varphi_i \in ((V^*)^q)^*$ as follows:

$$\varphi_i(f) := \sum_{j=1}^{q} f_j(h_i(v_j)), \qquad \forall f = (f_1, \ldots, f_q) \in (V^*)^q.$$

Since $h_1, \ldots, h_n$ are linearly independent, we must have that also $\varphi_1, \ldots, \varphi_n$ are linearly independent. That is because if $\lambda_1, \ldots, \lambda_n \in F$ are such that $\sum_{i=1}^{n} \lambda_i \varphi_i = 0$, then for all $1 \leq j \leq q$ and $g \in V^*$ we set $f = (f_1, \ldots, f_q) \in (V^*)^q$ with $f_j = g$ and $f_k = 0$ if $k \neq j$. Then

$$0 = \sum_{i=1}^{n} \lambda_i \varphi_i(f) = \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{q} f_j(h_i(v_j)) = \sum_{j=1}^{q} f_j \left(\sum_{i=1}^{n} \lambda_i h_i(v_j) \right) = g \left(\sum_{i=1}^{n} \lambda_i h_i(v_j) \right).$$

That is to say that $g\left(\sum_{i=1}^{n} \lambda_i h_i(v_j)\right) = 0$ for all $g \in V^*$. Therefore, we have $\sum_{i=1}^{n} \lambda_i h_i(v_j) = 0$ for all $1 \leq j \leq q$, and thus $\sum_{i=1}^{n} \lambda_i h_i = 0$. So, since $h_1, \ldots, h_n$ are linearly independent, $\lambda_i = 0$ for all i. Hence, $\varphi_1, \ldots, \varphi_n$ are linearly independent elements of $((V^*)^q)^*$.

Let W be the intersection of all kernels of φ_i, i.e., $W = \bigcap_{i=1}^{n} \mathrm{Ker}(\varphi_i)$, and consider the function:

$$\begin{aligned}\Phi\colon (V^*)^q/W &\to F^n\\ f+W &\mapsto (\varphi_1(f),\cdots,\varphi_n(f)).\end{aligned}$$

Φ is clearly injective and hence $\dim((V^*)^q/W) \leq n$. Moreover, as φ_i are linearly independent in $((V^*)^q)^*$ they are also linearly independent in $((V^*)^q/W)^*$. Then,

$$\dim((V^*)^q/W) = \dim(((V^*)^q/W)^*) = n.$$

In particular Φ is an isomorphism and there are elements $f_1,\dots,f_n \in (V^*)^q$ with $f_l = (f_{l1},\dots,f_{lq})$ for all $1 \leq l \leq n$, such that the matrix

$$(\varphi_i(f_l))_{1\leq i,l\leq n} = \left(\sum_{j=1}^{q} f_{lj}(h_i(v_j))\right)_{1\leq i,l\leq n}$$

has non-zero determinant.

The converse is clear, since if there exist elements $v_1,\dots,v_m$ in V and linear functionals $f_{jk} \in V^*$ for $1 \leq j \leq n$ and $1 \leq k \leq m$ such that the matrix

$$M = \left(\sum_{k=1}^{m} f_{jk}(h_i(v_k))\right)_{1\leq i,j\leq n}$$

has non-zero determinant, then for any $\lambda = (\lambda_1,\dots,\lambda_n) \in F^n$ with $\sum_{i=1}^{n} \lambda_i h_i = 0$ one has that $\lambda M = 0$. By the non-singularity of M, $\lambda = 0$ and then $h_1,\dots,h_n$ are linearly independent. □

Remark 3.2.8. *Let R be an integral domain. Let H be an R-algebra and M a left H-module. Then, for any maximal ideal $\mathfrak{m}$ of R, $\mathfrak{m}M$ is an R-submodule of M and $\mathfrak{m}H$ is an ideal of H.*

Now, using Lemmas 3.2.5 and 3.2.7, we are able to give a different proof of [13, Lemma 2.3].

Proposition 3.2.9. *Let R be an integral domain of characteristic 0 with $\mathrm{Jac}(R) = 0$ such that $\mathrm{char}(R/\mathfrak{m}) > 0$ for all maximal ideals $\mathfrak{m}$ of R. Let H be an R-algebra that is free of finite* *rank over R. For any faithful left H-module M that is free as an R-module and number $q > 1$, there exists a set of maximal ideals Y such that*

1. *$M/\mathfrak{m}M$ is a faithful $H/\mathfrak{m}H$-module for any $\mathfrak{m} \in Y$;*

2. $\mathrm{char}(R/\mathfrak{m}) > q$ *for any* $\mathfrak{m} \in Y$*;*

3. *the canonical homomorphism of R-algebras $\Psi : H_R \longrightarrow \prod_{\mathfrak{m}\in Y} H/\mathfrak{m}H$ is injective.*

Proof. Let R, H and M as in the statement of the Proposition. Since H is free of finte rank over R, say of rank n, then the H-action of H on M is given by n endomorphisms $h_i : M \to M$, for $i = 1, \dots, n$ (if $\{b_1, \dots, b_n\}$ is a basis for H, then we just set $h_i(m) := b_i m$, for all $m \in M$).

Since M is faithful, the elements $h_1, \dots, h_n$ are independent over R, in the sense that if $\sum_{i=1}^{n} r_i h_i = 0$ for some $r_1, \dots, r_n \in R$, then $r_1 = \cdots = r_n = 0$.

Let Q be the field of fractions of R and consider $M' = M \otimes_R Q$. Also, consider the Q-endomorphisms $h_i' = h_i \otimes id_Q : M' \to M'$, providing that the elements $h_1, \dots, h_n$ are independent over R, the elements $h_1', \dots, h_n'$ are linearly independent over Q. By Lemma 3.2.7, there exist elements $v_k \in M'$ and linear functions $f_{jk} : M' \to Q$ such that

$$0 \neq d = \det \left(\sum_{k=1}^{m} f_{jk}(h_i'(v_k)) \right)_{1 \leq i,j \leq n}.$$

Since Q is the field of fractions of R, it is possible to find $C \in R$ such that $w_k := C v_k \in M = M \otimes 1$ for all k. Let $\{b_\lambda \mid \lambda \in \Lambda\}$ be a basis for M as an R-module, then, since $\{h_i(w_k) \mid 1 \leq i \leq n \,\text{and}\, 1 \leq k \leq m\}$ is a finite set, there exists a finite subset $\Lambda' \subseteq \Lambda$ such that all elements $h_i(w_k)$ belong to the submodule spanned freely by b_λ for $\lambda \in \Lambda'$. Again, since Q is the field of fractions of R, there must exist a common denominator $D \in R$ of $f_{jk}(h_i(w_k))$ for all i, j, k. Define the R-linear maps $g_{jk} : M \to R$ by $g_{jk}(b_\lambda) = D f_{jk}(b_\lambda)$ for $\lambda \in \Lambda'$ and $g_{jk}(b_\lambda) = 0$ for $\lambda \in \Lambda \setminus \Lambda'$. Then

$$a := \det \left(\sum_{k=1}^{m} g_{jk}(h_i(w_k)) \right)_{1 \leq i,j \leq n} = C^n \det \left(\sum_{k=1}^{m} g_{jk}(h_i'(v_k)) \right)_{1 \leq i,j \leq n} = C^n D^n d \neq 0.$$

By Lemma 3.2.5, for any $q > 1$, the set $Y := X_{a,q}$ satisfies $\bigcap_{\mathfrak{m}\in Y} \mathfrak{m} = 0$.

(1) Let $\mathfrak{m} \in Y$ and $\pi : R \to R/\mathfrak{m}$ be the canonical projection. Consider $M/\mathfrak{m}M = M \otimes_R R/\mathfrak{m}$ and $H/\mathfrak{m}H = H \otimes_R R/\mathfrak{m}$. Set $\overline{m} = m + \mathfrak{m}M$ for all $m \in M$. By the induced $R/\mathfrak{m}$-linear maps

$$\overline{h_i} \colon M/\mathfrak{m}M \to M/\mathfrak{m}M \quad \text{given by} \quad \overline{m} \mapsto \overline{h_i(m)},$$

$H/\mathfrak{m}H$ acts on $M/\mathfrak{m}M$. Define $\overline{g_{jk}} = \pi \circ g_{jk}$ for all j, k, where g_{jk} are the R-linear maps from above.

Then $\overline{a} = \det \left(\sum_{k=1}^{m} \overline{g_{jk}}(\overline{h_i}(\overline{w_k})) \right)_{1 \leq i,j \leq n}$ is non-zero as $a \notin \mathfrak{m}$. By Lemma 3.2.7, $M/\mathfrak{m}M$ is a faithful $H/\mathfrak{m}H$-module.

(2) By definition, for any $\mathfrak{m} \in Y$, we have $\operatorname{char}(R/\mathfrak{m}) > q$.

(3) Since $\bigcap_{\mathfrak{m} \in Y} \mathfrak{m} = 0$, the canonical homomorphism $\pi_Y : R \hookrightarrow \prod_{\mathfrak{m} \in Y} R/\mathfrak{m}$ is injective [51, 3.12]. Tensoring with the free finite rank R-module H yields an injective (ring) homomorphism

$$1 \otimes \pi_Y : H_R \longhookrightarrow \prod_{\mathfrak{m} \in Y} H_R \otimes_R R/\mathfrak{m} = \prod_{\mathfrak{m} \in Y} H/\mathfrak{m}H.$$

□

3.3 Semisimple Hopf actions on Lie algebras and iterated Ore extensions

In this section we show that any action of a semisimple Hopf algebra on an enveloping algebra of a finite dimensional Lie algebra or on an interated Ore extension of derivation type factors through a group action. In a first step we will reduce our study of Hopf algebra over fields of characteristic zero to fields of positive characteristic.

3.3.1 Reduction to Hopf algebras over finite fields

Let R be an integral domain in characteristic 0 that is a Hilbert ring and with zero Jacobson radical.

Let H be a separable and coseparable Hopf algebra over R that is free of finite rank as an R-module. For any maximal ideal $\mathfrak{m}$ of R, consider the ideal $\mathfrak{m}H$ of H (Remark 3.2.8). Consider $H/\mathfrak{m}H = H_R \otimes_R F_\mathfrak{m}$, where $F_\mathfrak{m} = R/\mathfrak{m}$ is the finite field as in the previous section, then, since $H/\mathfrak{m}H \otimes_{F_\mathfrak{m}} H/\mathfrak{m}H = (H_R \otimes_R F_\mathfrak{m}) \otimes_{F_\mathfrak{m}} (H_R \otimes_R F_\mathfrak{m}) \cong H_R \otimes_R H_R \otimes_R F_\mathfrak{m}$, by defining

$$\Delta\colon H/\mathfrak{m}H \to H/\mathfrak{m}H \otimes_{F_\mathfrak{m}} H/\mathfrak{m}H \quad \text{via} \quad h \otimes 1_{F_\mathfrak{m}} \mapsto \Delta_R(h) \otimes 1_{F_\mathfrak{m}},$$

$\epsilon : H/\mathfrak{m}H \to F_\mathfrak{m}$ via $\epsilon(h \otimes 1_{F_\mathfrak{m}}) = \epsilon_R(h) 1_{F_\mathfrak{m}}$, and $S : H/\mathfrak{m}H \to H/\mathfrak{m}H$ by $S(h \otimes 1_{F_\mathfrak{m}}) = S_R(h) \otimes 1_{F_\mathfrak{m}}$, where Δ_R, ϵ_R and S_R are the comultiplication, counity and antipode of H_R respectively, $H/\mathfrak{m}H$ becomes a Hopf algebra over the finite field $F_\mathfrak{m}$.

Denote by $\overline{x}$ and $\overline{h}$ the elements in $R/\mathfrak{m}$ respectively $H/\mathfrak{m}H$. Since H is separable, by [34, 3.3], there exists a left integral $t \in H$ with $\epsilon(t) = 1$, which yields an integral $\overline{t} \in H/\mathfrak{m}H$ with $\epsilon(\overline{t}) = \overline{\epsilon(1)} = \overline{1}$. Hence, $H/\mathfrak{m}H$ is semisimple over $F_\mathfrak{m}$. Analogously, $(H/\mathfrak{m}H)^*$ is semisimple.

Given a left H-module algebra A that is free as an R-module, we extend the H-action on A to $A/\mathfrak{m}A$ by $\overline{h} \cdot \overline{a} = \overline{h \cdot a}$, for all $a \in A, h \in H$. With this action, $A/\mathfrak{m}A$ becomes a left $H/\mathfrak{m}H$-module algebra over a finite field $F_\mathfrak{m}$. This reduction from semisimple Hopf actions on a finitely presented algebra over a field of characteristic zero to an action of a semisimple and cosemisimple Hopf algebra over a finite field is the key step in [13]. In some cases, the algebras $A/\mathfrak{m}A$ over $F_\mathfrak{m}$ become finitely generated over their center. Extending the Hopf action to the skew-field of fractions of $A/\mathfrak{m}A$, when the degree of the skew-field of fractions and $\dim(H)!$ are coprime, Cuadra, Etingof and Walton showed that $H/\mathfrak{m}H$ has to be cocommutative. By Proposition 3.2.9(3), one concludes that H has to be cocommutative, and hence, by Corollary 2.2.39(ii), a group algebra if F is algebraically closed and so the action must factor through a group action.

As it was mentioned at the beginning of this chapter, in [18], Etingof and Walton showed that any semisimple Hopf action on a commutative domain A over an algebraically closed field must factor through a group action. In [18], Cuadra, Etingof and Walton extended this result in the following way:

Proposition 3.3.1 (Cuadra-Etingof-Walton, [13, Proposition 3.3])**.** *Let F be an algebraically closed field, H a semisimple, cosemisimple Hopf algebra over F acting inner faithfully on a division algebra D which is a finite module over its centre Z. If $[D : Z]$ and* $\dim(H)!$ *are coprime, then H is a group algebra.*

As a consequence of this Proposition and the reduction process to fields of positive characteristic, as described in section 3.2, one deduces:

Theorem 3.3.2 (Cuadra-Etingof-Walton). *Let H be a semisimple Hopf algebra over an algebraically closed field F of characteristic 0 acting on a finitely presented algebra A that is a Noetherian domain. Let R be the ring of structure constants of H and A as defined by ($\star$) and let H_R and A_R be the corresponding R-algebras. Suppose that there exists $q \geq 1$ such that for all maximal ideals $\mathfrak{m}$ of R with $\operatorname{char}(R/\mathfrak{m}) > q$ one has:*

- *the induced algebra $A_\mathfrak{m} = A_R \otimes_R R/\mathfrak{m}$ is a Noetherian domain;*
- *the skew-field of fractions $D_\mathfrak{m}$ of $A_\mathfrak{m}$ is finite over its center $Z_\mathfrak{m}$;*
- *$[D_\mathfrak{m} : Z_\mathfrak{m}]$ is coprime with $\dim(H)!$.*

Then the action of H on A factors through a group action.

Proof. Suppose that H acts inner faithfully on A. By Lemma 3.1.11, H acts faithfully on $A^{\otimes_F n}$ for some n. Passing from F to R, the faithfulness of the action is preserved, i.e., we also have that H_R acts faithfully on $A_R^{\otimes_R n}$. By Proposition 3.2.9, there exists a set Y of maximal ideals of R with $\operatorname{char}(R/\mathfrak{m}) > q$ and $A_R^{\otimes_R n} \otimes_R R/\mathfrak{m} = (A_R \otimes_R R/\mathfrak{m})^{\otimes_{R/\mathfrak{m}} n}$ being a faithful $H_\mathfrak{m} := H_R \otimes_R R/\mathfrak{m}$-module for all $\mathfrak{m} \in Y$. Again, by Lemma 3.1.11, $H_\mathfrak{m}$ acts inner faithfully on $A_\mathfrak{m} := A_R \otimes_R R/\mathfrak{m}$.

By assumption, the skew-field of fractions $D_\mathfrak{m}$ of $A_\mathfrak{m}$ is finite over its center and its dimension is coprime with $\dim(H)! = \dim(H_\mathfrak{m})!$. By [47, Theorem 2.2], the action of $H_\mathfrak{m}$ on $A_\mathfrak{m}$ extends to an action on $D_\mathfrak{m}$, which must be also inner faithful by Remark 3.1.9. It is easy to see that the same is true if we pass to the algebraic closure $\overline{R/\mathfrak{m}}$ of $R/\mathfrak{m}$ and tensor up $H_\mathfrak{m}, A_\mathfrak{m}$ and $D_\mathfrak{m}$.

By Proposition 3.3.1, $H_\mathfrak{m} \otimes_{R/\mathfrak{m}} \overline{R/\mathfrak{m}}$ is cocommutative and thus $H_\mathfrak{m}$ is cocommutative. By Proposition 3.2.9(3), the canonical R-algebra homomorphism

$$H_R \hookrightarrow \prod_{\mathfrak{m} \in Y} H_\mathfrak{m} = \prod_{\mathfrak{m} \in Y} H_R \otimes_R R/\mathfrak{m}$$

is injective. Since all $H_\mathfrak{m}$ are cocommutative, also H_R is cocommutative, and therefore H is as well. So H is a group algebra, by Corollary 2.2.39(ii), and then the theorem is proved. $\square$

3.3.2 Semisimple Hopf action on enveloping algebras of finite dimensional Lie algebras

As it was mentioned at the beginning of this chapter, it has already been outlined in [13, p.2] that the methods presented in Section 3.2 could be used to establish more general results on semisimple Hopf actions on quantized algebras. In these last two subsections of this chapter, we will show that that actions of semisimple Hopf algebras on enveloping algebras of finite-dimensional Lie algebras or on iterated differential operator rings over a field F of characteristic 0 factor through a group action. We start, in this subsection, by showing that this is the case for actions of semisimple Hopf algebras on enveloping algebras of finite-dimensional Lie algebras.

We recall that a *modular Lie algebra* is a Lie algebra over a field of positive characteristic (see [48]). For modular Lie algebras one has the following result from Farnsteiner and Strade's book [48, Chapter 6, Theorem 6.3(1)]:

Theorem 3.3.3 (Farnsteiner-Strade). *Let $U(\mathfrak{g})$ be the enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic p. Then the dimension of $\mathrm{Frac}(U(\mathfrak{g}))$ over its center is a power of p.*

This Theorem leads to the following corollary:

Corollary 3.3.4. *Any action of a semisimple Hopf algebra over an algebraically closed field F of characteristic zero on the enveloping algebra of a finite-dimensional Lie algebra factors through a group action.*

Proof. Let H be a semisimple Hopf algebra over a field F of characteristic 0. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over F and $A := U(\mathfrak{g})$ its universal enveloping algebra.

Suppose that H acts on A and denote by R the ring of structure constants of the action as in ($\star$). Using the structure constants of $\mathfrak{g}$, respectively H, define the Lie algebra $\mathfrak{g}_R$ over R of finite rank and the Hopf algebra H_R over R. Moreover, A_R is $U(\mathfrak{g}_R)$, the enveloping algebra of $\mathfrak{g}_R$ over the integral domain R.

Let $\mathfrak{m}$ be any maximal ideal of R, set $F_\mathfrak{m} = R/\mathfrak{m}$ and $p = \mathrm{char}(F_\mathfrak{m})$. Then

$$A_\mathfrak{m} := A_R \otimes_R F_\mathfrak{m} = U(\mathfrak{g}_R) \otimes_R F_\mathfrak{m} \simeq U(\mathfrak{g}_R \otimes_R F_\mathfrak{m})$$

is the enveloping algebra of the finite-dimensional Lie algebra $\mathfrak{g}_\mathfrak{m} := \mathfrak{g}_R \otimes_R F_\mathfrak{m}$ over the finite field $F_\mathfrak{m}$. By Theorem 3.3.3, one has $[D_\mathfrak{m} : Z_\mathfrak{m}] = p^m$, where $D_\mathfrak{m} = \mathrm{Frac}(A_\mathfrak{m})$ and $Z_\mathfrak{m} = Z(D_\mathfrak{m})$ for some $m \geq 0$. Hence for $q = \dim(H)!$ the assumptions of Theorem 3.3.2 are fulfilled and thus the Hopf algebra action of H on A factors through a group action. □

3.3.3 Iterated differential operator rings

In this last subsection, we will show that any action of semisimple Hopf algebras on the iterated Ore extensions of derivation type over polynomial rings factors through a group action.

Before we continue, we recall the definition of an Ore extension.

Definition 3.3.5. *Let S be a ring, α an automorphism of S, and δ an α-derivation on S. i.e., $\delta : S \to S$ is an additive map such that $\delta(rs) = \alpha(r)\delta(s) + \delta(r)s$, for all $r, s \in S$ (if $\alpha = id$, then δ is a derivation on S). We shall write $S' = S[x; \alpha, \delta]$ provided*

i) S' is a ring containing S as a subring;

ii) x is an element of S';

iii) S' is a free left S-module with basis $\{1, x, x^2, \ldots\}$;

iv) $xr = \alpha(r)x + \delta(r)$, for all $r \in S$.

We call S' a skew polynomial ring over S, or an Ore extension of S.

Given a commutative domain S, we say that an Ore extension of S is of *derivation type* if $\alpha = id$. By *iterated Ore extension*, we just mean that for an Ore extension of a ring S, say $S' = S[x_1; \alpha_1, \delta_1]$, we consider the Ore extension of S' and, via an iteration process of this type, for any natural number n, we get an iterated Ore extension $S[x_1; \alpha_1, d_1][x_2; \alpha_2, d_2][\cdots][x_n; \alpha_n, d_n]$. We call $S[x_1; \alpha_1, d_1][x_2; \alpha_2, d_2][\cdots][x_n; \alpha_n, d_n]$ an *iterated Ore extension of derivation type* over a commutative domain S, if all automorphisms α_i are the identity. For more on Ore extensions, see [21].

The Proposition below is the crucial step to show that the center of such iterated extensions of derivation type over a field of characteristic p is large.

Proposition 3.3.6. *Let A be a Noetherian domain over a field F of characteristic p. Suppose that A contains central elements $t_1, \ldots, t_n$, with $n \geq 1$, such that*

1. $B := F[t_1, \ldots, t_n]$ is a polynomial ring, and

2. A is a free B-module of rank p^m, for some $m \geq 0$.

Let d be any F-linear derivation of A. Then $A[x;d]$ contains $n+1$ central elements $\tilde{t}_1, \ldots, \tilde{t}_{n+1}$ such that $\widetilde{B} := F[\tilde{t}_1, \ldots, \tilde{t}_{n+1}]$ is a polynomial ring and $A[x;d]$ is free over $\widetilde{B}$ of rank p^{m+k} for some $k \geq 0$ depending on d.

Proof. If $d = 0$, then $\widetilde{B} = F[t_1, \ldots, t_n, x]$ is a central subring of $A[x]$ and $A[x]$ has rank p^m over $\widetilde{B}$. Thus, assume that $d \neq 0$. For any $a \in A$ we have $d(a^p) = 0$ as $\mathrm{char}(F) = p$ (see [21, p. 27] for a formula for a derivation on a ring). Thus $B' = F[t_1^p, \ldots, t_n^p]$ is a central subring of $A[x;d]$.

Using the hypothesis (2) we have that A has rank p^{n+m} over B'. Moreover, the derivation d is a B'-linear endomorphism of A and by the Cayley-Hamilton Theorem [6, 2.4], d will satisfy a monic polynomial $f \in B'[z]$, i.e., $f(d) \equiv 0$. For each number i, we divide z^{p^i} by the monic polynomial f. Then there are polynomials $q_i, r_i \in B'[z]$ such that $z^{p^i} = q_i f + r_i$ and $r_i = 0$ or $\deg(r_i) < \deg(f)$.

Since B' is Noetherian and r_k belongs to $B' \oplus \cdots \oplus B'z^m$, for $m = \deg(f)$, there must exist $k > 0$ such that $r_k \in \sum_{i=0}^{k-1} B' r_i$. Thus there are $a_{k-1}, \ldots, a_1 \in B'$ such that $r_k = a_0 r_0 + \cdots + a_{k-1} r_{k-1}$. Hence define

$$g := z^{p^k} - \sum_{i=0}^{k-1} a_i z^{p^i} = \left(q_k - \sum_{i=0}^{k-1} a_i q_i \right) f.$$

As f is a factor of g, we also have $g(d) \equiv 0$. Set $\Theta := x^{p^k} - \sum_{i=0}^{k-1} a_i x^{p^i} \in B'[x] \subset A[x;d]$. Note that Θ commutes with powers of x, since the coefficients of Θ are central in $A[x;d]$. Furthermore, let $a \in A$, then

$$\Theta a - a\Theta = d^{p^k}(a) - \sum_{i=0}^{k-1} a_i d^{p^i}(a) = g(d)(a) = 0.$$

Hence Θ is central in $A[x;d]$. Since Θ is monic and of positive degree in x, we have that Θ and $t_1^p, \ldots, t_n^p$ are algebraically independent over F. Thus they form a central subring $B'[\Theta] = F[t_1^p, \ldots, t_n^p, \Theta]$ of $A[x;d]$. As $A[x;d]$ has rank p^k over $A[\Theta]$ and since $A[\Theta]$ has rank

p^{n+m} over $B'[\Theta]$, we conclude that $A[x;d]$ has rank p^{n+m+k} over $B'[\Theta]$, and the proposition is proved. □

Note that if B is a Noetherian central subring of a Noetherian domain A such that A is finitely generated over B, then $D := \mathrm{Frac}(A)$ can be obtained by inverting the elements of B (see [38, Theorem 13.6.5] or [8, I.13.3]). Hence if A is free of rank p^n over B, then $[D : \mathrm{Frac}(B)] = p^n$. In particular, $[D : Z]$ is a power of p, where Z denotes the center of D.

Finally, we get the following corollary.

Corollary 3.3.7. *Any action of a semisimple Hopf algebra H over an algebraically closed field F of characteristic zero on an iterated Ore extension of derivation type over a polynomial ring in finitely many variables factors through a group action.*

Proof. Let H be a semisimple Hopf algebra over F. Let $A = S[x_1; d_1][x_2; d_2][\cdots][x_n; d_n]$ be an iterated Ore extension of derivation type over a polynomial ring S in finitely many variables. We may assume $S = F$. Suppose that H acts on A and let R be the ring of structure constants of H and A as in $(\star)$. Then we can consider the R-Hopf algebra H_R acting on the R-algebra $A_R = R[x_1; d_1][x_2; d_2][\cdots][x_n; d_n]$.

For any maximal ideal $\mathfrak{m}$ of R we set $F_\mathfrak{m} = R/\mathfrak{m}$ and $p = \mathrm{char}(F_\mathfrak{m})$. Then $A_\mathfrak{m} = A_R \otimes_R F_\mathfrak{m} = F_\mathfrak{m}[x_1; d_1][x_2; d_2][\cdots][x_n; d_n]$ is an iterated Ore extension of derivation type over $F_\mathfrak{m}$, a field of positive characteristic. By Proposition 3.3.6, $A_\mathfrak{m}$ contains a central subring B, which is a polynomial ring, such that $A_\mathfrak{m}$ is free over B with rank a power of p. By the argument right before this corollary, $D_\mathfrak{m} = \mathrm{Frac}(A_\mathfrak{m})$ has a p-power as dimension over its center. Thus, for $q = \dim(H)!$, by Theorem 3.3.2, we have that the Hopf algebra action of H on A factors through a group action. □

Corollary 3.3.7 covers the case of the nth Weyl algebra over a polynomial ring, i.e. [13, Proposition 4.3], but also other examples like the Jordan plane $A = \mathbb{C}[x][y; x^2\frac{\partial}{\partial x}]$, which is a generalization of [10, Theorem 0.1] as the assumption that the H-action preserves the filtration of A can be removed.

During all this chapter, the assumption of H being a semisimple Hopf algebra was needed. It is worth to mention that recently, in [14], the authors proved a similar result with the assumption of H being only finite-dimensional and not necessarily semisimple.

3.4 A remark on smash and crossed products

In this section, we shall make a brief comment on smash and crossed products.

Recall the definitions of smash and crossed products, Definitions 2.3.8 and 2.3.10. Before we continue, we present the following definition and theorems.

Definition 3.4.1 ([39, Definition 7.2.1])**.** *Let $A \subset B$ be algebras, and H be a Hopf algebra.*

1) *$A \subset B$ is a (right) H-extension if B is a right H-comodule algebra with $B^{coH} = A$.*

2) *The H-extension $A \subset B$ is H-cleft if there exists a right H-comodule map $\gamma : H \to B$ which is convolution invertible.*

Theorem 3.4.2 ([39, Theorem 7.2.2])**.** *An H-extension $A \subset B$ is H-cleft if and only if $B \cong A\#_\sigma H$.*

Theorem 3.4.3 ([17, Theorem 7.2.11])**.** *Let H be a finite dimensional Hopf algebra, R a normal Hopf subalgebra of H, and $\overline{H} = H/HR^+$ the associated factor Hopf algebra. Then H is isomorphic to a certain crossed product $R\#_\sigma\overline{H}$ as algebras.*

Having in mind these two theorems, we are able to prove the following one.

Theorem 3.4.4. *Let H be a semisimple Hopf algebra, A an algebra, and R a normal Hopf subalgebra of H. Suppose also that H acts on A. Then $A\#H$ is isomorphic to a certain crossed product $(A\#R)\#_\sigma\overline{H}$ as algebras, where $\overline{H} = H/HR^+$.*

Proof. Let $R \subseteq H$ be a normal Hopf subalgebra of H. Then $I = HR^+ = R^+H$ is a Hopf ideal of H. Also, it is not difficult to see that H becomes a right $\overline{H} = H/I$-comodule algebra via $(id \otimes \pi)\Delta : H \to H \otimes \overline{H}$, where π is the projection map from H to $\overline{H}$.

Define $\rho : A\#H \to (A\#H) \otimes \overline{H}$ via $a\#h \mapsto \sum(a\#h_1) \otimes \overline{h_2}$. This makes $A\#H$ an $\overline{H}$-comodule algebra. Moreover, for any $r \in R$, $\pi(r) = \epsilon(r)\overline{1_H}$,

$$\rho(a\#r) = \sum(a\#r_1) \otimes \overline{r_2} = \sum(a\#r_1) \otimes \epsilon(r_2)\overline{1_H} = (a\#r) \otimes \overline{1_H}.$$

That is, $A\#R \subseteq (A\#H)^{co\overline{H}}$. Since ${}_RR$ is injective (R is a Frobenius Algebra since $\dim(R) < \infty$), the inclusion map $i : R \hookrightarrow H$ splits, i.e., there exists a left R-module map

$\theta : H \to R$, with $\theta i = id_R$. We call the composition $i\theta = \varphi$. Let $Q : H \to H$ be defined as $h \mapsto \sum S(h_1)\varphi(h_2)$. So, for all $r \in R$ and $h \in H$,

$$\begin{aligned} Q(rh) &= \sum S(h_1)S(r_1)(\varphi(r_2h_2)) \\ &= \sum S(h_1)S(r_1)r_2\varphi(h_2) \\ &= \epsilon(r)\sum S(h_1)\varphi(h_2) = \epsilon(r)Q(h). \end{aligned}$$

Thus, $Q(R^+H) = 0$ and then, by the isomorphism theorem for vector spaces, there exists a linear map $\overline{Q} : \overline{H} \to H$ such that $\overline{Q}\pi = Q$. Since $\sum h_1Q(h_2) = \sum h_1S(h_2)\varphi(h_3) = \varphi(h)$, we deduce that $\sum h_1\overline{Q}\pi(h_2) = \varphi(h)$. Let $a\#h \in (A\#H)^{co\overline{H}}$. Then, $\sum(a\#h_1)\otimes\overline{h_2} = (a\#h)\otimes\overline{1_H}$ and so

$$a\#\varphi(h) = \sum a\#h_1\overline{Q}\pi(h_2) = a\#h\overline{Q}(\overline{1_H}) = a\#h.$$

Therefore, identifying $i(R)$ with its image in H, we have $A\#R = (A\#H)^{co\overline{H}}$. This entails that $A\#R \subseteq A\#H$ is a (right) $\overline{H}$-extension.

By Theorems 3.4.3 and 3.4.2, we conclude that $R \subseteq H$ is an $\overline{H}$-extension and $\overline{H}$-cleft. Hence, there exists a convolution invertible morphism of $\overline{H}$-comodules $\gamma : \overline{H} \to H$.

Define $\overline{\gamma} : \overline{H} \to A\#H$ via $\overline{h} \mapsto 1_A\#\gamma(\overline{h})$. Then $\overline{\gamma}$ is an $\overline{H}$-comodule morphism. Indeed, since γ is a morphism of $\overline{H}$-comodules, the following diagram is commutative:

$$\begin{tikzcd} \overline{H} \arrow[r, "\gamma"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowleft"] & H \arrow[d, "(id\otimes\pi)\Delta"] \\ \overline{H}\otimes\overline{H} \arrow[r, "\gamma\otimes Id"'] & H\otimes\overline{H}. \end{tikzcd}$$

Then, $\rho(\overline{\gamma}(\overline{h})) = \sum(1_A\#\gamma(\overline{h})_1)\otimes\overline{\gamma(\overline{h})_2} = \sum(1_A\#\gamma(\overline{h_1}))\otimes\overline{h_2} = (\overline{\gamma}\otimes id)\Delta(\overline{h})$, i.e., the diagram

$$\begin{tikzcd} \overline{H} \arrow[r, "\overline{\gamma}"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowleft"] & A\#H \arrow[d, "\rho"] \\ \overline{H}\otimes\overline{H} \arrow[r, "\overline{\gamma}\otimes Id"'] & (A\#H)\otimes\overline{H} \end{tikzcd}$$

is commutative and so $\overline{\gamma}$ is an $\overline{H}$-comodule morphism.

Also, define $\psi : \overline{H} \to A\#H$ via $\overline{h} \mapsto 1_A\#\gamma^{-1}(\overline{h})$, where γ^{-1} is the convolution inverse of γ. For the same reasons as before, ψ is an $\overline{H}$-comodule map. Moreover, for all $h \in H$,

$$\begin{aligned}(\overline{\gamma} * \psi)(\overline{h}) &= \sum \overline{\gamma}(h_1)\psi(h_2)\\ &= \sum (1_A\#\gamma(\overline{h_1}))(1_A\#\gamma^{-1}(\overline{h_2}))\\ &= \sum 1_A\#\gamma(\overline{h_1})\gamma^{-1}(\overline{h_2})\\ &= \epsilon(\overline{h})1_A\#1_H\end{aligned}$$

and

$$\begin{aligned}(\psi * \overline{\gamma})(\overline{h}) &= \sum \psi(h_1)\overline{\gamma}(h_2)\\ &= \sum (1_A\#\gamma^{-1}(\overline{h_1}))(1_A\#\gamma(\overline{h_2}))\\ &= \sum 1_A\#\gamma^{-1}(\overline{h_1})\gamma(\overline{h_2})\\ &= \epsilon(\overline{h})1_A\#1_H.\end{aligned}$$

This entails that $\overline{\gamma}$ is a convolution invertible $\overline{H}$-comodule map. So, $A\#R \subseteq A\#H$ is $\overline{H}$-cleft. Then, by Theorem 3.4.2, $A\#H \cong (A\#R)\#_\sigma\overline{H}$. □

Suppose that I is a Hopf ideal of the form $I = HR^+$ for some normal Hopf subalgebra R of H. If I is contained in $\mathrm{Ann}_H(A)$, then the action of R on A is trivial, since $\epsilon(r)1_H - r \in I$ for all $r \in R$. Hence, $A\#H \cong (A \otimes R)\#_\sigma H/I$. In particular, if H is semisimple, then R and H/I are semisimple [45, Corollary 8.43]. If, moreover, A is semiprime and $\mathrm{char}(F) = 0$, then $A \otimes R$ is semiprime (see [28, Theorem 3.1]). Thus if H/I is strongly semiprime (i.e, it answers Cohen's question in the affirmative, see [35]), e.g. if H/I is a group ring, then $A\#H$ is semiprime.

Chapter 4

Semisimple Hopf actions which do not factor through group actions

In the last chapter, we mentioned that Etingof and Walton, in [18], have shown that any semisimple Hopf action over an algebraically closed field on a commutative domain must factor through a group action. Also in the last chapter, we extend their result by showing that any action of a semisimple Hopf algebra H on an enveloping algebra of a finite-dimensional Lie algebra or on an iterated Ore extension of derivation type factors through a group action. Let the ground field F be of characteristic zero. In this chapter, we will present actions of semisimple Hopf algebras over an algebraically closed field of characteristic zero which do not factor through group actions.

In order to do that, we will construct a class of semisimple Hopf algebras H_{2n^2}, which are not group algebras, and show that there exist inner faithful actions of those algebras on the quantum polynomial algebras.

4.1 Twisting of a Hopf algebra and skew polynomial rings

In Chapter 2, Example 2.2.21, we presented the Hopf algebra H_8, which was discovered by Kac and Paljutkin in the 1960's. It is a non-commutative, non-cocommutative and, since $S^2 = id$, by Theorem 2.2.37, a semisimple Hopf algebra of dimension 8. Later, in 1995, Masouka has shown that there is only one (up to isomorphisms) semisimple Hopf algebra of dimension 8 that is neither commutative nor cocommutative (see [37]). Although Masouka presents H_8 under the perspective of biproducts and bicrossed products, H_8 can also be presented as a quotient of a certain skew polynomial ring.

In this section, in order to justify this mentioned presentation of H_8, and to construct a family of Hopf algebras, we will give conditions to extend the structure of a given Hopf algebra R to the quotient $R[z,\sigma]/I$, where $R[z;\sigma]$ is the *skew polynomial ring of automorphism type* of R and I is a bi-ideal of $R[z,\sigma]$. We start with the definition of a twist for a bialgebra.

Definition 4.1.1 ([42, Definition 7.8.1])**.** *Let R be a bialgebra and J be an invertible element in $R\otimes R$. J is called a right twist (or a Drinfel'd twist) for R if J satisfies:*

(i) $(id\otimes\Delta)(J)(1\otimes J)=(\Delta\otimes id)(J)(J\otimes 1)$;

(ii) $(id\otimes\epsilon)(J)=1=(\epsilon\otimes id)(J)$.

J is called a left twist for R if it satisfies (ii) and $(1\otimes J)(id\otimes\Delta)(J)=(J\otimes 1)(\Delta\otimes id)(J)$.

If J is a left twist for R, then J^{-1} is a right twist for R ([42, Lemma 7.8.2]).

Definition 4.1.2 ([15, 2.1])**.** *Let R be a bialgebra. Let J be a right twist for R and $\sigma\in\operatorname{End}(R)$. We say that the pair (σ,J) is a twisted homomorphism for R if σ satisfies:*

(i) $J(\sigma\otimes\sigma)\Delta(h)=\Delta(\sigma(h))J$ *for all* $h\in R$;

(ii) $\epsilon\circ\sigma=\epsilon$.

Note that, for any homomorphism of coalgebras $\sigma\in\operatorname{End}(R)$, the pair $(\sigma,1\otimes 1)$ is a twisted homomorphism for R.

Remark 4.1.3. *Let R be a bialgebra, (σ,J) a twist homomorphism for R and $a\in R$.*

1. *If σ is an automorphism, then* $(\sigma\otimes\sigma)\Delta(\sigma^{-1}(a))=J^{-1}\Delta(a)J:=\Delta^J(a)$;
2. *Since J is invertible,* $(\sigma\otimes\sigma)\Delta(a)=J^{-1}\Delta(\sigma(a))J=\Delta^J(\sigma(a))$;
3. *σ is an homomorphism of coalgebras if and only if J commutes with $\Delta(\sigma(a))$, for all $a\in R$.*

Recalling the Definition 3.3.5, given a ring S we say that the skew polynomial ring (Ore extension) $S[x;\alpha,\delta]$ is of *automorphism type* if $\delta=0$. Also, we will need the following proposition.

Proposition 4.1.4 ([21, Proposition 2.4])**.** *Let $S = R[x;\alpha,\delta]$ be a skew polynomial ring. Suppose that we have a ring T, a ring homomorphism $\phi : R \to T$, and an element $y \in T$ such that $y\phi(r) = \phi(\alpha(r))y + \phi(\delta(r))$ for all $r \in R$. Then there is a unique ring homomorphism $\psi : S \to T$ such that $\psi_{|R} = \phi$ and $\psi(x) = y$.*

Note that, for an Ore extension of automorphism type, the condition to extend the homomorphism ϕ is given by the existence of an element $y \in T$ such that

$$y\phi(r) = \phi(\alpha(r))y, \tag{4.1}$$

for all $r \in R$.

Now, using these notions, we shall extend the bialgebra structure of a bialgebra R to its Ore extension of automorphism type $R[z;\sigma]$, for some specific $\sigma \in \mathrm{End}(R)$.

Theorem 4.1.5. *Let R be a bialgebra and (σ, J) be a twisted homomorphism for R. Let $H = R[z;\sigma]$ be the skew polynomial ring of automorphism type. Then the bialgebra structure of R can be extended to H such that $\Delta(z) = J(z \otimes z)$ and $\epsilon(z) = 1_F$. Conversely, if there exist an invertible element $J \in R \otimes R$ and $\sigma \in \mathrm{Aut}(R)$ such that $R[z,\sigma]$ is a bialgebra with $\Delta(z) = J(z \otimes z)$ and $\epsilon(z) = 1$, then (σ, J) is a twisted homomorphism for R.*

Proof. Let (σ, J) be a twisted homomorphism for the bialgebra R, and let $H = R[z;\sigma]$ be the Ore extension of automorphism type of R. Since R is a bialgebra, $\Delta : R \to R \otimes R$ is a homomorphism of algebras. As $R \hookrightarrow H$, then we have a homomorphism of algebras $\Delta : R \to H \otimes H$. Consider the element $J(z \otimes z) \in H \otimes H$. Then, by condition (i) on Definition 4.1.2, for all $h \in R$, we have

$$J(z \otimes z)\Delta(h) = J(\sigma \otimes \sigma)\Delta(h)(z \otimes z) = \Delta(\sigma(h))J(z \otimes z).$$

Thus Δ satisfies the Ore condition as in 4.1. So, by Proposition 4.1.4, there exists a unique algebra homomorphism $\overline{\Delta} : H \to H \otimes H$ such that $\overline{\Delta}_{|R} = \Delta$ and $\overline{\Delta}(z) = J(z \otimes z)$. While it may be an abuse of notation, we just write $\overline{\Delta} = \Delta$.

Furthermore,

$$\begin{aligned}(id\otimes\Delta)\Delta(z) &= (id\otimes\Delta)(J)(z\otimes\Delta(z))\\ &= (id\otimes\Delta)(J)(1\otimes J)(z\otimes z\otimes z)\\ &\overset{(\star)}{=} (\Delta\otimes id)(J)(J\otimes 1)(z\otimes z\otimes z)\\ &= (\Delta\otimes id)(J)(\Delta(z)\otimes z)\\ &= (\Delta\otimes id)\Delta(z);\end{aligned}$$

where in $(\star)$ we are using the condition (i) of Definition 4.1.1. Note that the extension in Proposition 4.1.4 is unique. Hence the two maps $(\Delta\otimes id)\Delta$ and $(id\otimes\Delta)\Delta$ have to coincide, since z has the same image under both of them. Therefore, $\Delta : H \to H\otimes H$ is a coassociative map.

Now, since $\epsilon : R \to F$ is a homomorphism of algebras, then, using the condition (ii) of Definition 4.1.2, for all $h \in R$, we have

$$1_F\epsilon(h) = 1_F\epsilon(\sigma(h)) = \epsilon(\sigma(h))1_F.$$

Thus ϵ satisfies the Ore condition as in 4.1. So, by Proposition 4.1.4, there exists a unique algebra homomorphism $\bar{\epsilon} : H \to F$ such that $\bar{\epsilon}_{|_R} = \epsilon$ and $\bar{\epsilon}(z) = 1_F$. Again, while it may be an abuse of notation, we just write $\bar{\epsilon} = \epsilon$. Moreover, using the condition (ii) of the Definition 4.1.1, we have

$$(id\otimes\epsilon)\Delta(z) = (id\otimes\epsilon)(J)z = z = (\epsilon\otimes id)(J)z = (\epsilon\otimes id)\Delta(z).$$

Here, again, note that the extension in Proposition 4.1.4 is unique and then the two maps $(id\otimes\epsilon)\Delta$ and $(\epsilon\otimes id)\Delta$ have to coincide, since z has the same image under both of them. Thus ϵ satisfies the counity property in H. Therefore, the bialgebra structure of R extends to H as stated in the lemma.

Now, to prove the converse, suppose that there exist an invertible element $J \in R\otimes R$ and $\sigma \in \mathrm{Aut}(R)$ such that $R[z,\sigma]$ is a bialgebra with $\Delta(z) = J(z\otimes z)$ and $\epsilon(z) = 1$. Note that since $(id\otimes\epsilon)\Delta(z) = z = (\epsilon\otimes id)\Delta(z)$, we must have that $(id\otimes\epsilon)(J) = 1 = (\epsilon\otimes id)(J)$. Also, note that

$$(id \otimes \Delta)\Delta(z) = (id \otimes \Delta)(J)(z \otimes \Delta(z)) = (id \otimes \Delta)(J)(1 \otimes J)(z \otimes z \otimes z)$$

and

$$(\Delta \otimes id)\Delta(z) = (\Delta \otimes id)(J)(\Delta(z) \otimes z) = (\Delta \otimes id)(J)(J \otimes 1)(z \otimes z \otimes z).$$

Since $R[z, \sigma]$ is a bialgebra, we have that $(id \otimes \Delta)\Delta(z) = (\Delta \otimes id)\Delta(z)$ and hence $(id \otimes \Delta)(J)(1 \otimes J) = (\Delta \otimes id)(J)(J \otimes 1)$, that is, J is a right twist for R. Moreover, for all $h \in R$, $zh = \sigma(h)z$. This implies that

$$\epsilon(h) = \epsilon(zh) = \epsilon(\sigma(h)z) = \epsilon(\sigma(h)),$$

i.e., $\epsilon \circ \sigma = \epsilon$. Also, note that

$$J(\sigma \otimes \sigma)\Delta(h)(z \otimes z) = J(z \otimes z)\Delta(h) = \Delta(zh) = \Delta(\sigma(h)z) = \Delta(\sigma(h))J(z \otimes z), \quad \forall h \in R.$$

Thus, $J(\sigma \otimes \sigma)\Delta(h) = \Delta(\sigma(h))J$ and hence the pair (σ, J) is a twisted homomorphism for R. □

As it was said at the beginning of this section, given a Hopf algebra R, we will find conditions to define a Hopf algebra structure on the quotient $R[z, \sigma]/I$, for some bi-ideal I of $R[z, \sigma]$. The following lemma gives us certain conditions to find the bi-ideal on the bialgebra $R[z; \sigma]$ which will be used to define such Hopf algebra structure.

Lemma 4.1.6. *Let R be a bialgebra and (σ, J) be a twisted homomorphism for R. Suppose that there exists $0 \neq t \in R$ such that $\Delta(t) = J(\sigma \otimes \sigma)(J)(t \otimes t)$. Then, for $H = R[z; \sigma]$ with the bialgebra structure as in Theorem 4.1.5, $I = \langle z^2 - t \rangle$ is a bi-ideal of H.*

Proof. Let R be a bialgebra and (σ, J) be a twisted homomorphism for R. By Theorem 4.1.5, $H = R[z; \sigma]$ is also a bialgebra. Let $0 \neq t \in R$ as in the hypothesis and let I be the ideal in H generated by $z^2 - t$. We have to prove that $I = \langle z^2 - t \rangle$ is coideal of H. We note that t

necessarily satisfies

$$\begin{aligned}
t &= (\epsilon \otimes id)\Delta(t) \\
&= (\epsilon \otimes id)(J)(\epsilon \otimes id)(\sigma \otimes \sigma)(J)\epsilon(t)t \\
&= (\epsilon \otimes id)(\sigma \otimes \sigma)(J)\epsilon(t)t \\
&= (\epsilon \otimes \sigma)(J)\epsilon(t)t \\
&= \sigma((\epsilon \otimes id)(J))\epsilon(t)t \\
&= \sigma(1)\epsilon(t)t = \epsilon(t)t,
\end{aligned}$$

which implies $\epsilon(t) = 1$. So, $\epsilon(z^2 - t) = \epsilon(z)^2 - \epsilon(t) = 0$. That is, $I = \langle z^2 - t\rangle \subseteq Ker(\epsilon)$.

Furthermore,

$$\begin{aligned}
\Delta(z^2 - t) &= \Delta(z)^2 - \Delta(t) \\
&= J(z \otimes z)J(z \otimes z) - \Delta(t) \\
&= J(\sigma \otimes \sigma)(J)(z^2 \otimes z^2) - \Delta(t) \\
&= J(\sigma \otimes \sigma)(J)(z^2 - t \otimes z^2) + J(\sigma \otimes \sigma)(J)(t \otimes z^2 - t) + J(\sigma \otimes \sigma)(J)(t \otimes t) - \Delta(t) \\
&= J(\sigma \otimes \sigma)(J)(z^2 - t \otimes z^2) + J(\sigma \otimes \sigma)(J)(t \otimes z^2 - t),
\end{aligned}$$

which belongs to $I \otimes H + H \otimes I$. Therefore, I is a bi-ideal of H. □

Remark 4.1.7. *Note that, conversely, if $I = \langle z^2 - t\rangle$ is a bi-ideal of H, then $\Delta(t) - J(\sigma \otimes \sigma)(J)(t \otimes t) \in H \otimes I + I \otimes H$.*

Hence, given a bialgebra R and (σ, J) a twisted homomorphism for R and an element t that satisfies the hypothesis of Lemma 4.1.6, we have that H/I is a bialgebra, for $H = R[z; \sigma]$ and $I = \langle z^2 - t\rangle$. The next lemma presents conditions to extend a Hopf algebra structure from R to the quotient bialgebra H/I.

Lemma 4.1.8. *Let R be a Hopf algebra with antipode S, (σ, J) be a twisted homomorphism, and $\sigma \circ S = S \circ \sigma$ with the additional condition that $\sigma^2 = id$. Suppose that there exists $0 \neq t \in R$ such that $\Delta(t) = J(\sigma \otimes \sigma)(J)(t \otimes t)$. If $S(t) = t$ and*

(i) $tJ^1S(J^2) = 1$;

(ii) $t\sigma(S(J^1)J^2) = 1$,

where $J = J^1 \otimes J^2$ *with the summation omitted, then there exists a Hopf algebra structure on* H/I *with* $S(z) = z$*. Conversely, if there exists a Hopf algebra structure on* H/I *with* $S(z) = z$*, then* $tJ^1S(J^2) = 1 = t\sigma(S(J^1)J^2)$*.*

Proof. Let R be a Hopf algebra with antipode S and (σ, J) a twisted homomorphism for R such that $\sigma^2 = id$. By Theorem 4.1.5, $R[z,\sigma]$ is a bialgebra, and by Lemma 4.1.6, $I = \langle z^2 - t\rangle$ is a bi-ideal of $R[z,\sigma]$.

By abuse of notation, we just write h for the element $h + I$ of H/I. And to define the antipode, we just extend the antipode S of R to H/I defining $S(z) = z$. We note that $S : H/I \to H/I$ is well defined, since $S(z^2 - t) = S(z)^2 - S(t) = z^2 - t \in I$ and, using that $S \circ \sigma = \sigma \circ S$, $S(a)z = zS(\sigma(a))$ for all $a \in R$. Also, we have that

$$\mu(id \otimes S)\Delta(z) = \mu(id \otimes S)(J(z \otimes z)) = J^1zS(z)S(J^2) = z^2\sigma^2(J^1)S(J^2) = tJ^1S(J^2) = 1,$$

and

$$\mu(S\otimes id)\Delta(z) = \mu(S\otimes id)(J(z\otimes z)) = S(z)S(J^1)J^2z = z^2\sigma(S(J^1))\sigma(J^2) = t\sigma(S(J^1)J^2) = 1.$$

Since S is an antipode for R, the antipode property is verified for R as well. Therefore, S is an antipode of H/I and so H/I is a Hopf algebra. The converse follows from the two equations above. □

In this setting, for R a semisimple Hopf algebra, we have the following corollary.

Corollary 4.1.9. *Under the conditions of Lemma 4.1.8, if* R *is a semisimple Hopf algebra, then* H/I *is semisimple.*

Proof. Note that since R is semisimple, by Remark 2.2.38, R is finite-dimensional. Hence, H/I is finite-dimensional. Also, by Theorem 2.2.37, $S^2 = id$ on R. Now, since $S(z) = z$, we conclude that $S^2 = id$ on H/I. Thus, again by Theorem 2.2.37, H/I is semisimple. □

4.2 Semisimple Hopf algebras of dimension $2n^2$

In this section, using what we have done in the last section, we shall construct semisimple Hopf algebras of dimension $2n^2$, which, in the sequel, will be used to define actions on the quantum plane which do not factor through group actions. From now on, we let the ground field F be algebraically closed.

Before we properly start this section, we recall that for an algebra A, an element $e \in A$ is called *idempotent* if $e^2 = e$. If $e_1, e_2 \in A$ are idempotents we call them *orthogonal* if $e_1e_2 = 0 = e_2e_1$. A finite set of orthogonal idempotents $\{e_1, \cdots, e_m\}$ is called *complete* if $e_1 + \cdots + e_m = 1_A$.

Let $\Gamma = \langle x \mid x^n = 1\rangle$ be the cyclic group of order $n > 1$. Let $q \in F$ be a primitive nth root of unity. For every integer j, we set

$$e_j = \frac{1}{n}\sum_{i=0}^{n-1} q^{-ij}x^i.$$

Observe that if $j \equiv j'(\text{mod } n)$, then $q^j = q^{j'}$ and $x^j = x^{j'}$, and therefore $e_j = e_{j'}$. This means that $e_0, \ldots, e_{n-1}$ lists the distinct $e_i's$. Moreover, for $0 \leq j, k < n$, we have

$$e_jx^k = \frac{1}{n}\sum_{i=0}^{n-1} q^{-ij}x^{i+k} = q^{jk}\left(\frac{1}{n}\sum_{i=0}^{n-1} q^{-(i+k)j}x^{i+k}\right) = q^{jk}e_j. \tag{4.2}$$

For the next lemma, recall that for any nth root of unity $q \neq 1$, we have $\sum_{i=0}^{n-1} q^i = 0$, since $0 = q^n - 1 = (q-1)\sum_{i=0}^{n-1} q^i$.

Lemma 4.2.1. $\{e_0, \cdots, e_{n-1}\}$ *is a complete set of orthogonal idempotents of* $F[\Gamma]$.

Proof. Since q^{-j} is also an nth root of unity different from 1 if $j \neq 0$, we get

$$\sum_{i=0}^{n-1} e_i = \frac{1}{n}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} q^{-ij}x^j = \frac{1}{n}\sum_{j=0}^{n-1}\left(\sum_{i=0}^{n-1}\left(q^{-j}\right)^i\right)x^j = 1,$$

Also, using (4.2), for $0 \leq l, j < n$:

$$e_je_l = \frac{1}{n}\sum_{k=0}^{n-1} q^{-lk}e_jx^k = \frac{1}{n}\sum_{k=0}^{n-1} q^{-lk+jk}e_j = \frac{1}{n}\sum_{k=0}^{n-1}\left(q^{j-l}\right)^k e_j = \begin{cases} e_j & \text{if } l = j \\ 0 & \text{if } l \neq j \end{cases}$$

Hence, $\{e_0, \cdots, e_{n-1}\}$ is a complete set of orthogonal idempotents of $F[\Gamma]$. $\square$

We denote the elements of $G = \Gamma \times \Gamma$ by $x^i y^s$ for $0 \leq i, s < n$. Let $\{e_0, \cdots, e_{n-1}\}$ be the complete set of idempotents of $F[\Gamma]$ as in Lemma 4.2.1. Let $\sigma \in \mathrm{Aut}(F[G])$ be the automorphism of $F[G]$ induced by the group isomorphism

$$x^i y^s \mapsto x^s y^i, \quad \forall 1 \leq i, s \leq n.$$

Set $\overline{e_i} := \sigma(e_i)$, i.e., $\overline{e_i} = \frac{1}{n}\sum_{j=0}^{n-1} q^{-ij} y^j$. As in equation (4.2) one has $\overline{e_i} y^k = q^{ik}\overline{e_i}$.

Now, in $F[G] \otimes F[G]$ consider the element

$$J := \sum_{i=0}^{n-1} e_i \otimes y^i.$$

Note that we can also write J in terms of the elements $\overline{e_i}$'s as

$$J = \frac{1}{n}\sum_{i,j=0}^{n-1} q^{-ij} x^j \otimes y^i = \sum_{i=0}^{n-1} x^i \otimes \overline{e_i}. \tag{4.3}$$

With this setting, we have the following lemma.

Lemma 4.2.2. *The pair (σ, J) is a twisted homomorphism for $F[G]$.*

Proof. First we note that J is invertible with inverse $J^{-1} = \sum_{j=0}^{n-1} e_j \otimes y^{-j}$. Using (4.2) and (4.3), we get

$$\begin{aligned}
(\Delta \otimes 1)(J)(J \otimes 1) &= \left(\sum_{i=0}^{n-1} \Delta(x^i) \otimes \overline{e_i}\right)\left(\sum_{j=0}^{n-1} e_j \otimes y^j \otimes 1\right) \\
&= \sum_{i,j=0}^{n-1} x^i e_j \otimes x^i y^j \otimes \overline{e_i} \\
&= \sum_{i,j=0}^{n-1} q^{ij} e_j \otimes y^j x^i \otimes \overline{e_i} \\
&= \sum_{i,j=0}^{n-1} e_j \otimes y^j x^i \otimes y^j \overline{e_i} \\
&= \left(\sum_{j=0}^{n-1} e_j \otimes \Delta(y^j)\right)\left(\sum_{j=0}^{n-1} 1 \otimes x^i \otimes \overline{e_i}\right) = (1 \otimes \Delta)(J)(1 \otimes J).
\end{aligned}$$

Moreover, we have

$$(1 \otimes \epsilon)(J) = \sum_{i=0}^{n-1} e_i = 1 = \sum_{i=0}^{n-1} \overline{e_i} = (\epsilon \otimes 1)(J).$$

That is, J is a right twist as in Definition 4.1.1. For $x^k y^s \in G$, note that

$$(\sigma \otimes \sigma)\Delta(x^k y^s) = (\sigma \otimes \sigma)(x^k y^s \otimes x^k y^s) = x^s y^k \otimes x^s y^k = \Delta(\sigma(x^k y^s)).$$

Thus, since $F[G]$ is commutative, we have that $J(\sigma \otimes \sigma)\Delta(x^k y^s) = \Delta(\sigma(x^k y^s))J$. Moreover, clearly $\epsilon \circ \sigma = \epsilon$. Therefore, the pair (σ, J) is a twisted homomorphism as in Definition 4.1.2. □

Hence, by Theorem 4.1.5, $H = F[G][z;\sigma]$ is bialgebra with $\Delta(z) = J(z \otimes z)$ and $\epsilon(z) = 1$. Now, consider the element $t = \sum_{i=0}^{n-1} e_i y^i$ which satisfies

$$\sigma(t) = \sum_{i=0}^{n-1} \overline{e_i} x^i = \frac{1}{n} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} q^{-ij} y^j x^i = \sum_{j=0}^{n-1} \left(\frac{1}{n} \sum_{i=0}^{n-1} q^{-ij} x^i \right) y^j = \sum_{j=0}^{n-1} e_j y^j = t.$$

Moreover, t has an inverse in $F[G]$, $t^{-1} = \sum_{i=0}^{n-1} e_i y^{-i}$, because $\left(\sum_{i=0}^{n-1} e_i y^i \right) \left(\sum_{k=0}^{n-1} e_k y^{-k} \right) = \sum_{i,k=0}^{n-1} e_i e_k y^{i-k} = \sum_{i=0}^{n-1} e_i y^0 = 1$, since the e_i's form a complete set of orthogonal idempotents.

Lemma 4.2.3. *The element t satisfies $\Delta(t) = J(\sigma \otimes \sigma)(J)(t \otimes t)$.*

Proof. Since $\sum_{k=0}^{n-1} (q^{(l-j)})^k = 0$ for $l \neq j$, note that for any i

$$\begin{aligned}
\sum_{k=0}^{n-1} e_k \otimes e_{(i-k)} &= \frac{1}{n^2} \sum_{k,j,l=0}^{n-1} q^{-jk} q^{-l(i-k)} x^j \otimes x^l \\
&= \frac{1}{n^2} \sum_{k,j,l=0}^{n-1} q^{(l-j)k} q^{-li} x^j \otimes x^l \\
&= \frac{1}{n^2} \sum_{j,l=0}^{n-1} \left(\sum_{k=0}^{n-1} (q^{(l-j)})^k \right) q^{-li} x^j \otimes x^l \\
&= \frac{1}{n} \sum_{j=0}^{n-1} q^{-ij} x^j \otimes x^j = \Delta(e_i).
\end{aligned}$$

So, it follows that $\Delta(t) = \sum_{i,m=0}^{n-1} e_i y^m \otimes e_{(m-i)} y^m$. Then, by (4.3), we get

$$\begin{aligned}
J(\sigma \otimes \sigma)(J)(t \otimes t) &= \sum_{i,j=0}^{n-1} e_i y^j t \otimes y^i e_j t \\
&= \sum_{i,j,k,l=0}^{n-1} e_i e_k y^{j+k} \otimes e_j e_l y^{i+l} \\
&= \sum_{i,j=0}^{n-1} e_i y^{i+j} \otimes e_j y^{i+j} \\
&= \sum_{i,m=0}^{n-1} e_i y^m \otimes e_{(m-i)} y^m = \Delta(t).
\end{aligned}$$

□

So, t satisfies the hypothesis of Lemma 4.1.6 and hence $I = \langle z^2 - t \rangle$ is a bi-ideal of $F[G][z;\sigma]$. Thus $H = F[G][z;\sigma]/\langle z^2 - t \rangle$ is also a bialgebra.

Note that $\sigma^2 = id$ and

$$S(t) = \frac{1}{n} \sum_{i,j=0}^{n-1} q^{-ij} x^{-i} y^{-j} = \frac{1}{n} \sum_{k,s=0}^{n-1} q^{-ks} x^k y^s = \sum_{s=0}^{n-1} e_s y^s = t.$$

Since $\sigma(S(x^k y^s)) = x^{-s} y^{-k} = S(\sigma(x^k y^s))$ and

$$tJ^{(1)} S(J^{(2)}) = \sum_{i,j=0}^{n-1} e_i e_j y^{i-j} = \sum_{i=0}^{n-1} e_i = 1,$$

and

$$t\sigma(S(J^{(1)})J^{(2)}) = t\sigma\left(\frac{1}{n}\sum_{i,j=0}^{n-1} q^{-ij} x^{-j} y^i\right) = t\sum_{j=0}^{n-1} y^{-j} \frac{1}{n} \sum_{i=0}^{n-1} q^{-ij} x^i = t\left(\sum_{j=0}^{n-1} e_j y^{-j}\right) = tt^{-1} = 1,$$

by Lemma 4.1.8, $H = F[G][z;\sigma]/\langle z^2 - t \rangle$ is a Hopf algebra.

Note that these Hopf algebras have dimension $2n^2$ and we shall denote them by H_{2n^2}.

Also, we note that since F has characteristic zero, by the comment made right after Theorem 2.2.36, $F[G]$ is semisimple, and then, by Corollary 4.1.9, H_{2n^2} is also semisimple. Moreover, H_{2n^2} is non-commutative and non-cocommutative, since $\Delta(z) \neq \tau(\Delta(z))$, where $\tau(\sum h_1 \otimes h_2) = \sum h_2 \otimes h_1$.

Before we continue, we establish the following lemma, which will be useful in the sequel.

Lemma 4.2.4. *Let I be a Hopf ideal of H_{2n^2} and consider $R = F[G]$. If $I \cap R = 0$, then $I = 0$.*

Proof. Let I be a Hopf ideal of H_{2n^2} such that $I \cap R = 0$. Consider the restriction of the projection map $\pi_{|_R} : R \to H_{2n^2}/I$. Clearly, $\operatorname{Ker} \pi_{|_R} = I \cap R$. Hence, we can look at R as a Hopf subalgebra of H_{2n^2}/I. Then, by Corollary 2.2.44, $\dim(R)$ divides $\dim(H_{2n^2}/I)$.

Since $\dim(H_{2n^2}) = 2\dim(R)$, we must have that $\dim(H_{2n^2}/I) = 2\dim(R)$ or $\dim(H_{2n^2}/I) = \dim(R)$. If $\dim(H_{2n^2}/I) = 2\dim(R)$, then $I = 0$.

If $\dim(H_{2n^2}/I) = \dim(R)$, then $H_{2n^2}/I \cong R$ and thus H_{2n^2}/I is commutative. Hence, we must have that $\bar{x}\bar{z} = \bar{z}\bar{x} = \bar{y}\bar{z}$, which implies that $(x - y)z \in I$. So, $(x - y)z(zt^{-1}) = (x - y) \in I \cap R = 0$, which is absurd. $\square$

4.2.1 Non-cocommutative extensions of finite abelian groups

The construction we have made for H_{2n^2} can also be made for G a finite abelian group. In order to justify this, we expose the following about the tensor product of extensions we considered in Section 4.1.

Let A and B be bialgebras. Suppose that we can find (α, W) and (β, L) twisted homomorphisms of A and B respectively. Then, by Theorem 4.1.5, the Ore extensions $A[z_1; \alpha]$ and $B[z_2; \beta]$ become bialgebras with $\Delta_A(z_1) = W(z_1 \otimes z_1)$ and $\epsilon(z_1) = 1$, and $\Delta_B(z_2) = L(z_2 \otimes z_2)$ and $\epsilon(z_2) = 1$ respectively.

From now until the end of this exposition for tensor products, consider $R = A \otimes B$ with the usual tensor product bialgebra structure. Define $J = (id \otimes \tau_{A,B} \otimes id)(W \otimes L)$, where $\tau_{A,B}$ is the twist map. Let $\sigma : R \to R$ be defined as $\sigma = \alpha \otimes \beta$.

With this setting, we have the following lemma.

Lemma 4.2.5. *(σ, J) as defined above is a twisted homomorphism for the tensor product bialgebra $R = A \otimes B$.*

Proof. Write $J = (W^{(1)} \otimes L^{(1)}) \otimes (W^{(2)} \otimes L^{(2)})$, where $W = W^{(1)} \otimes W^{(2)}$ and $L = L^{(1)} \otimes L^{(2)}$ with the summation omitted. Since (α, W) and (β, L) are twisted homomorphism, the relations of Definitions 4.1.1 and 4.1.2 are satisfied for (α, W) and (β, L). That is to say that we have the following equations

$$\sum W^{(1)} \otimes W_1^{(2)} W^{(1)'} \otimes W_2^{(2)} W^{(2)'} = \sum W_1^{(1)} W^{(1)'} \otimes W_2^{(1)} W^{(2)'} \otimes W^{(2)}, \tag{4.4}$$

$$W^{(1)} \epsilon_A(W^{(2)}) = 1_A = \epsilon_A(W^{(1)})(W^{(2)}), \tag{4.5}$$

$$\sum W^{(1)} \alpha(h_1) \otimes W^{(2)} \alpha(h_2) = \sum (\alpha(h))_1 W^{(1)} \otimes (\alpha(h))_2 W^{(2)}, \quad \forall h \in A, \tag{4.6}$$

$$\epsilon_A \circ \alpha = \epsilon_A. \tag{4.7}$$

For L the equations are analogous. Then, using these relations, we get

$$\begin{aligned}
(id_R \otimes \Delta_R)(J)(1 \otimes J) &= \sum W^{(1)} \otimes L^{(1)} \otimes W_1^{(2)} W^{(1)'} \otimes L_1^{(2)} L^{(1)'} \otimes W_2^{(2)} W^{(2)'} \otimes L_2^{(2)} L^{(2)'} \\
&\overset{(\star)}{=} \sum W_1^{(1)} W^{(1)'} \otimes L_1^{(1)} L^{(1)'} \otimes W_2^{(1)} W^{(2)'} \otimes L_2^{(1)} L^{(2)'} \otimes W^{(2)} \otimes L^{(2)} \\
&= (\Delta_R \otimes id_R)(J)(J \otimes 1),
\end{aligned}$$

where in $(\star)$ we are using (4.4). Moreover, by (4.5), we have that $(id_R \otimes \epsilon_R)(J) = W^{(1)} \epsilon_A(W^{(2)}) \otimes L^{(1)} \epsilon_B(L^{(2)}) = 1_A \otimes 1_B$. By the same reasons, $(\epsilon_R \otimes id_R)(J) = 1_A \otimes 1_B$.

Now, for $h \in A$ and $g \in B$, using (4.6), we have

$$\begin{aligned}
J(\sigma \otimes \sigma)\Delta_R(h \otimes g) &= \sum W^{(1)} \alpha(h_1) \otimes L^{(1)} \beta(g_1) \otimes W^{(2)} \alpha(h_2) \otimes L^{(2)} \beta(g_2) \\
&= \sum (\alpha(h))_1 W^{(1)} \otimes (\beta(g))_1 L^{(1)} \otimes (\alpha(h))_2 W^{(2)} \otimes (\beta(g))_2 L^{(2)} \\
&= \Delta_R(\sigma(h \otimes g))J.
\end{aligned}$$

Clearly, because of (4.7), we have that $\epsilon_R \circ \sigma = \epsilon_R$. Therefore, (σ, J) is a twisted homomorphism for R. □

Hence, by Lemma 4.2.5 and Theorem 4.1.5, the Ore extension $R[z, \sigma]$ is a bialgebra with $\Delta(z) = J(z \otimes z)$ and $\epsilon(z) = 1$.

Now, suppose that there exist elements $a \in A$ and $b \in B$ satisfying the hypothesis of Lemma 4.1.6, i.e.,

$$\Delta_A(a) = W(\alpha \otimes \alpha)(W)(a \otimes a) \quad \text{and} \quad \Delta_B(b) = L(\beta \otimes \beta)(L)(b \otimes b). \tag{4.8}$$

Lemma 4.2.6. *Let $t = a \otimes b \in R$. Then $\Delta_R(t) = J(\sigma \otimes \sigma)(J)(t \otimes t)$.*

Proof. Indeed, by (4.8), we have

$$\begin{aligned}\Delta_R(t) &= \sum a_1 \otimes b_1 \otimes a_2 \otimes b_2 \\ &= W^{(1)}\alpha(W^{(1)'})a \otimes L^{(1)}\beta(L^{(1)'})b \otimes W^{(2)}\alpha(W^{(2)'})a \otimes L^{(2)}\beta(L^{(2)'})b \\ &= J(\sigma \otimes \sigma)(J)(t \otimes t).\end{aligned}$$

□

Therefore, by Lemma 4.1.6, $\langle z^2 - t\rangle$ is a bi-ideal of $R[z;\sigma]$.

Suppose now that A and B are Hopf algebras, with antipode S_A and S_B respectively. So the tensor product Hopf algebra $R = A \otimes B$ has antipode $S = S_A \otimes S_B$. Suppose also that A and B satisfy the hypothesis of Lemma 4.1.8. That is to say that $S_A(a) = a$, $\alpha^2 = id$, $S_A \circ \alpha = \alpha \circ S_A$,

$$aW^{(1)}S_A(W^{(2)}) = 1, \quad a\alpha(S_A(W^{(1)})W^{(2)}) = 1, \tag{4.9}$$

$S_B(b) = b$, $\beta^2 = id$, $S_B \circ \beta = \beta \circ S_B$,

$$bL^{(1)}S_B(L^{(2)}) = 1 \quad \text{and} \quad b\beta(S_B(L^{(1)})L^{(2)}) = 1. \tag{4.10}$$

Clearly, $S(t) = t$. Moreover, $\sigma^2 = id$. Also, by (4.9) and by (4.10), we have

$$\begin{aligned}tJ^1S(J^2) &= (a \otimes b)(W^{(1)} \otimes L^{(1)})(S_A(W^{(2)}) \otimes S_B(L^{(2)})) \\ &= aW^{(1)}S_A(W^{(2)}) \otimes bL^{(1)}S_B(L^{(2)}) \\ &= 1 \otimes 1\end{aligned}$$

and

$$\begin{aligned} t\sigma(S(J^1)J^2) &= (a\otimes b)(\alpha\otimes\beta)((S_A(W^{(1)})\otimes S_B(L^{(1)}))(W^{(2)}\otimes L^{(2)})) \\ &= a\alpha(S_A(W^{(1)})W^{(2)})\otimes b\beta(S_B(L^{(1)})L^{(2)}) \\ &= 1\otimes 1. \end{aligned}$$

So, by Lemma 4.1.8, $R[z;\sigma]/\langle z^2-t\rangle$ is a Hopf algebra.

By an induction argument, we have the following proposition.

Proposition 4.2.7. *Let $A_1,\dots,A_n$, for some positive integer n, be bialgebras with twisted homomorphisms (α_i, J_i) and elements a_i satisfying all the conditions necessary to make $A_i[z_i,\alpha_i]/\langle z_i^2-a_i\rangle$ a Hopf algebra. Then $R[z;\sigma]/\langle z^2-t\rangle$ is a Hopf algebra, for $R = A_1\otimes\cdots\otimes A_n$.*

Remark 4.2.8. *Let G be a finite abelian group of order n and $q\in F$ be a primitive nth root of unity. The construction of H_{2n^2} can be made not only for cyclic groups. It can also be made for abelian groups and in this remark we first justify this by using what we have done above.*

Since G is a finite abelian group, then $G\cong G_1\times\cdots\times G_s$, where, for $1\le j\le s$, $G_j=\langle g_j \mid g_j^{n_j}=1\rangle$ is a cyclic group of order n_j.

Let $\Gamma = G\times G\cong (G_1\times G_1)\times(G_2\times G_2)\times\cdots\times(G_s\times G_s)$. Then,

$$F[\Gamma]\cong F[G_1\times G_1]\otimes F[G_2\times G_2]\otimes\cdots\otimes F[G_s\times G_s].$$

For each Hopf algebra $F[G_i\times G_i]$, for $1\le i\le s$, we can apply the construction we have made for H_{2n^2} and conclude that there exists (σ_i, J_i) a twisted homomorphism for $F[G_i\times G_i]$ and an element $t_i\in F[G_i\times G_i]$ such that $F[G_i\times G_i][z_i,\sigma_i]/\langle z_i^2-t_i\rangle$ is a Hopf algebra. Hence, by Proposition 4.2.7, for $R=F[G_1\times G_1]\otimes F[G_2\times G_2]\otimes\cdots\otimes F[G_s\times G_s]$, we can construct a twisted homomorphism (σ, J) for R and an element $t\in R$ such that $R[z,\sigma]/\langle z^2-t\rangle$ is a Hopf algebra.

Explicitly, using a set of complete orthogonal idempotents for $F[G]$ constructed by Radford in [43, Section 4], the Hopf algebra structure can be obtained as follows. Let q be a primitive nth root of unity and set $q_i = q^{n/n_i}$.

Let $\Lambda = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_s}$*. For* $\mathbf{m} \in \Lambda$*,* $\mathbf{m} = (m_1, \cdots, m_s)$*, we define*

$$g^{\mathbf{m}} := (g_1^{m_1}, \cdots, g_s^{m_s}) \text{ and } q^{\mathbf{m}} := q_1^{m_1} \cdots q_s^{m_s}$$

Λ *is isomorphic to* G *via* $\mathbf{m} \mapsto g^{\mathbf{m}}$*. Then, for all* $\mathbf{m} \in \Lambda$*, we define*

$$e_{\mathbf{m}} = \frac{1}{n} \sum_{\mathbf{r} \in \mathbf{\Lambda}} q^{-\mathbf{rm}} g^{\mathbf{r}},$$

where $(q^{\mathbf{r}})^{\mathbf{m}} = q^{\mathbf{rm}}$*.*

One verifies that the set $\{e_{\mathbf{m}}\}_{\mathbf{m} \in \mathbf{\Lambda}}$ *is a complete set of orthogonal idempotents of* $F[G]$*. Despite the abuse of notation, denote the elements of* $G \times G$ *by* $g^{\mathbf{r}} h^{\mathbf{s}}$ *for* $\mathbf{r}, \mathbf{s} \in \Lambda$*. The automorphism* $\sigma \in \mathrm{Aut}(F[G \times G])$ *is given by* $g^{\mathbf{r}} h^{\mathbf{s}} \mapsto g^{\mathbf{s}} h^{\mathbf{r}}$ *and the element* J *is equal to* $J := \sum_{\mathbf{r} \in \Lambda} e_{\mathbf{r}} \otimes h^{\mathbf{r}}$*. Then* (σ, J) *is a twisted homomorphism and so, by Theorem 4.1.5, the bialgebra structure of* $F[G \times G]$ *can be extended to the Ore extension* $F[G \times G][z; \sigma]$ *with* $\Delta(z) = J(z \otimes z)$ *and* $\epsilon(z) = 1$*.*

The element $t = \sum_{\mathbf{r} \in \Lambda} e_{\mathbf{r}} h^{\mathbf{r}}$ *satisfies* $\sigma(t) = t$*,* $t^{-1} = \sum_{\mathbf{r} \in \Lambda} e_{\mathbf{r}} h^{-\mathbf{r}}$*, and, as in Lemma 4.2.3,* $\Delta(t) = J(\sigma \otimes \sigma)(J)(t \otimes t)$*. So, by Lemma 4.1.6,* $I = \langle z^2 - t \rangle$ *is a bi-ideal of* $F[G \times G][z; \sigma]$ *and thus* $H = F[G \times G][z; \sigma]/\langle z^2 - t \rangle$ *is also a bialgebra.*

Also, note that $\sigma^2 = id$*,* $S(t) = t$*,* $tJ^1 S(J^2) = 1$*, and* $t\sigma(S(J^1)J^2) = 1$*. Moreover,* $\sigma \circ S = S \circ \sigma$ *holds. Hence, by Lemma 4.1.8,* $H = F[G \times G][z; \sigma]/\langle z^2 - t \rangle$ *is a Hopf algebra with* $S(z) = z$*.*

4.3 Semisimple Hopf algebra of dimension 8

In the Example 2.2.21, we presented the Kac and Paljutkin's Hopf algebra of dimension 8 as the algebra generated by x, y, and z subject to some relations. In this section, we will present H_8 as the Hopf algebra H_{2n^2}, for $n = 2$. Also, using such a presentation of H_8, we shall classify its Hopf ideals.

4.3.1 H_8 as a quotient of an Ore extension

First, we recall that in Example 2.2.21, H_8 was introduced as the algebra generated by x, y, and z subject to the following relations

$$x^2 = 1, \quad y^2 = 1, \quad xy = yx$$
$$z^2 = \frac{1}{2}(1 + x + y - xy), \quad zx = yz, \quad zy = xz,$$

and with coalgebra structure given by

$$\begin{aligned} \Delta(x) &= x \otimes x, & \epsilon(x) &= 1 \\ \Delta(y) &= y \otimes y, & \epsilon(y) &= 1 \\ \Delta(z) &= \tfrac{1}{2}(1 \otimes 1 + x \otimes 1 + 1 \otimes y - x \otimes y)(z \otimes z), & \epsilon(z) &= 1. \end{aligned}$$

And the antipode is given by $S(x) = x$, $S(y) = y$, and $S(z) = z$.

Now, H_8 can also be viewed as a Hopf algebra as the ones constructed in Section 4.2. For the Hopf algebras constructed in that Section, we take $n = 2$ and $q = -1$. The group Γ, then, is the cyclic group of order 2, $\mathbb{Z}_2 = \{x \mid x^2 = 1\}$. In this setting, we have that the orthogonal idempotents of $F[\mathbb{Z}_2]$ as in Lemma 4.2.1 are given by:

$$e_0 = \frac{1}{2}(1 + x) \text{ and } e_1 = \frac{1}{2}(1 - x).$$

Then, for $G = \langle x, y \mid x^2 = 1 = y^2, xy = yx \rangle = \Gamma \times \Gamma$, the automorphism σ swaps x and y, i.e., $\sigma(x) = y$ and $\sigma(y) = x$. And the element J is given by

$$J = \frac{1}{2}((1 + x) \otimes 1 + (1 - x) \otimes y) = \frac{1}{2}(1 \otimes 1 + x \otimes 1 + 1 \otimes y - x \otimes y).$$

Let $R = F[G]$. So, $R[z; \sigma]$ becomes a bialgebra with

$$\Delta(z) = J(z \otimes z) = \frac{1}{2}(1 \otimes 1 + x \otimes 1 + 1 \otimes y - x \otimes y)(z \otimes z),$$

and $\epsilon(z) = 1$. Also, note that $zx = \sigma(x)z = yz$. Since t is given by

$$t = e_0 + e_1 y = \frac{1}{2}(1 + x + y - xy),$$

we get that $z^2 = \frac{1}{2}(1 + x + y - xy)$ in the Hopf algebra $R[z;\sigma]/\langle z^2 - t\rangle$, where $S(z) = z$.

So, H_{2n^2}, for $n = 2$, is precisely the Hopf algebra H_8 as introduced in Example 2.2.21. Then, from now on, every time we refer to H_8, we keep in mind its presentation as the one presented in this section, i.e., as a quotient of an Ore extension: $H_8 = R[z;\sigma]/\langle z^2 - t\rangle$.

4.3.2 Hopf ideals of H_8

Hereafter, let $F = \mathbb{C}$, the field of complex numbers. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ be the Klein four group. Keeping in mind the presentation of H_8 as in the last subsection, i.e., $H_8 = R[z;\sigma]/\langle z^2 - t\rangle$, for $R = \mathbb{C}[G]$, our goal in this subsection is to classify the Hopf ideals of H_8.

Let J be a Hopf ideal of H_8. Consider $I = J \cap R$. By Lemma 2.2.13, I is a Hopf ideal of R. Since $R = \mathbb{C}[G]$ is a group algebra, by Lemma 3.1.7, there exists a normal subgroup N of $G = \langle x, y : x^2 = 1 = y^2, xy = yx\rangle$, such that $I = R\mathbb{C}[N]^+$. Since G is abelian, all subgroups of G are normal and the list of subgroups of G is the following one:

$$\langle x\rangle, \langle y\rangle, \langle xy\rangle, \{1\} \text{ and } G.$$

So, in order to classify all the Hopf ideals J of H_8, we can separate in cases for $N \in \{\langle x\rangle, \langle y\rangle, \langle xy\rangle, \{1\}, G\}$ and $I = R\mathbb{C}[N]^+$.

Theorem 4.3.1. *Any Hopf ideal of H_8 is one of the following:* (0)*,* $H_8 \ker(\epsilon_{|_R})$*,* $\ker(\epsilon)$*,* $H_8(x - y)$*,* $H_8(x - y) + H_8(1 - (\lambda xz + \bar{\lambda} z))H_8$*,* $J = H_8(x - y) + H_8(1 - (\lambda z + \bar{\lambda} xz))H_8$*, where* $\lambda = \frac{1+i}{2}$ *and* $\overline{\lambda}$ *is its complex conjugate.*

In the sequel, we will prove this theorem in a chain of lemmas. We start with the case $N = G$.

Lemma 4.3.2. *If $N = G$, then $J = H_8 \ker(\epsilon_{|_R})$ or $J = \ker(\epsilon)$.*

Proof. If $N = G$, then $I = \ker(\epsilon_{|_R})$. Note that for $a \in I$, $az = z\sigma(a)$ and so $\epsilon(\sigma(a)) = 0$, i.e., $\sigma(a) \in \ker(\epsilon_{|_R})$ and so, $az \in H_8 I$ for all $a \in I$. This implies that $H_8 I$ is a Hopf ideal of H_8 and then, since $H_8 I \subseteq J$, we have that $J/H_8 I$ is a Hopf ideal of $H_8/H_8 I$.

Note that $(1-x), (1-y), (1-xy) \in H_8 I$, so, every element $\bar{h} \in H_8/H_8 I$, for $\bar{h} = h + H_8 I$, can be written as $\bar{h} = \alpha\bar{1} + \beta\bar{z}$, for some $\alpha, \beta \in \mathbb{C}$. We conclude that $H_8/H_8 I$ is generated as vector space by 1 and z. Therefore, $\dim(H_8/H_8 I) \leq 2$ and hence $\dim(J/H_8 I) \leq 2$. This gives us three options, or $\dim(J/H_8 I) = 0$, or $\dim(J/H_8 I) = 1$, or $\dim(J/H_8 I) = 2$.

If $\dim(J/H_8I) = 0$, then $J = H_8I$.

Now, in order to argue for the two remain cases, note that $\{1-x, 1-y, 1-xy, z-xz, z-yz, z-xyz\}$ is a linearly independent set in H_8I and so, $\dim(H_8I) \geq 6$. Thus, if $\dim(J/H_8I) = 1$, then $\dim(J) \geq 7$. But $J \neq H_8$ and then we get that J has codimension 1. This and the fact that $J \subseteq \ker(\epsilon)$ and $\ker(\epsilon)$ has codimension 1 lead us to conclude that $J = \ker(\epsilon)$.

Finally, $\dim(J/H_8I)$ cannot be 2. Otherwise $J = H_8$, and then $R = \ker(\epsilon_{|_R})$, which is absurd.

Therefore, for $N = G$, either $J = H_8 \ker(\epsilon_{|_R})$ or $J = \ker(\epsilon)$.

□

Corollary 4.3.3. *If x belongs to N, then $J = H_8 \ker(\epsilon_{|_R})$ or $J = \ker(\epsilon)$.*

Proof. Suppose that $x \in N$. Then, $1 - x \in R\mathbb{C}[N]^+ = I \subseteq J$. Since $t^2 = 1$ in H_8 and J is an ideal,

$$z(1-x)zt = (1-y)z^2t = (1-y)t^2 = (1-y) \in J.$$

Thus $1 - y \in J \cap R = R\mathbb{C}[N]^+$. Hence, y must belong to N, otherwise we would have that $N = \langle x \rangle$ and then every $h \in \mathbb{C}[N]^+$ is written as $h = \beta 1 - \beta x$, for some $\beta \in \mathbb{C}$. Thus, since $(1-y) \in R\mathbb{C}[N]^+$, we should have that

$$(1-y) = (\alpha_1 1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy)(\beta 1 - \beta x),$$

for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \in \mathbb{C}$. But this leads to a contradiction. Therefore, $y \in N$ and hence $N = G$. By Lemma 4.3.2, $J = H_8 \ker(\epsilon_{|_R})$ or $J = \ker(\epsilon)$.

The case $y \in N$ is analogous. □

This corollary is saying that N cannot be $\langle x \rangle$ or $\langle y \rangle$. So, it remains the cases where $N = \langle 1 \rangle$ or $N = \langle xy \rangle$.

Lemma 4.3.4. *If $N = \langle 1 \rangle$, then $J = 0$.*

Proof. If $N = \langle 1 \rangle$, then $\mathbb{C}[N]^+ = 0$, i.e., $0 = I = R \cap J$. By Lemma 4.2.4, $J = 0$.

□

Lemma 4.3.5. *If $N = \langle xy \rangle$, then $J = H_8(x-y)$ or $J = H_8(x-y) + H_8(1-(\lambda xz + \bar{\lambda} z))H_8$ or $J = H_8(x-y) + H_8(1-(\lambda z + \bar{\lambda} xz))H_8$, for $\lambda = \dfrac{1+i}{2}$ and $\bar{\lambda}$ its complex conjugate.*

Proof. Note that if $N = \langle xy \rangle$, then $\mathbb{C}(x-y) = I = R\mathbb{C}[N]^+$. Also, since $(x-y)z = -z(x-y) \in H_8 I$, we can conclude that $H_8 I$ is a Hopf ideal of H_8.

Note that, in the quotient H_8/H_8I, every element r of R can be written as $\bar{r} = \alpha\bar{1} + \beta\bar{x}$, for some $\alpha, \beta \in \mathbb{C}$, and then, since every element of H_8 can be written as $h = a + bz$, for $a, b \in R$, in the quotient we have that $\bar{h} = \alpha_1\bar{1} + \alpha_2\bar{x} + \alpha_3\bar{z} + \alpha_4\bar{x}\bar{z}$, for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{C}$.

So, for $\overline{H} = H_8/H_8I$, we have that $\dim(\overline{H}) \leq 4$. Now, consider the elements $\bar{1}$, $\bar{x}$, $\bar{g}$, and $\bar{h}$, where $\bar{g} = \lambda\bar{x}\bar{z} + \bar{\lambda}\bar{z}$, and $\bar{h} = \bar{\lambda}\bar{x}\bar{z} + \lambda\bar{z}$, for $\lambda = \dfrac{1+i}{2}$ and $\bar{\lambda} = \dfrac{1-i}{2}$. We claim that $\bar{1}, \bar{x}, \bar{g}, \bar{h}$ are distinct group-likes elements. Indeed, $\Delta(\bar{1}) = \bar{1}\otimes\bar{1}$ and $\Delta(\bar{x}) = \bar{x}\otimes\bar{x}$. For $\bar{g}$ and $\bar{h}$, note first that $\Delta(\bar{z}) = \frac{1}{2}(\bar{z}\otimes\bar{z} + \bar{x}\bar{z}\otimes\bar{z} + \bar{z}\otimes\bar{x}\bar{z} - \bar{x}\bar{z}\otimes\bar{x}\bar{z})$ and $\Delta(\bar{x}\bar{z}) = \frac{1}{2}(\bar{x}\bar{z}\otimes\bar{x}\bar{z} + \bar{z}\otimes\bar{x}\bar{z} + \bar{x}\bar{z}\otimes\bar{z} - \bar{z}\otimes\bar{z})$. Hence,

$$
\begin{aligned}
\Delta(\bar{g}) &= \lambda\Delta(\bar{x}\bar{z}) + \bar{\lambda}\Delta(\bar{z}) \\
&= \frac{1}{2}\left((\bar{\lambda}-\lambda)\bar{z}\otimes\bar{z} + (\bar{\lambda}+\lambda)\bar{x}\bar{z}\otimes\bar{z} + (\bar{\lambda}+\lambda)\bar{z}\otimes\bar{x}\bar{z} + (\lambda-\bar{\lambda})\bar{x}\bar{z}\otimes\bar{x}\bar{z}\right) \\
&= \frac{1}{2}\left(-i\bar{z}\otimes\bar{z} + \bar{x}\bar{z}\otimes\bar{z} + \bar{z}\otimes\bar{x}\bar{z} + i\bar{x}\bar{z}\otimes\bar{x}\bar{z}\right) \\
&= \bar{\lambda}\bar{z}\otimes\bar{\lambda}\bar{z} + \lambda\bar{x}\bar{z}\otimes\bar{\lambda}\bar{z} + \bar{\lambda}\bar{z}\otimes\lambda\bar{x}\bar{z} + \lambda\bar{x}\bar{z}\otimes\lambda\bar{x}\bar{z} \\
&= \lambda\bar{x}\bar{z} + \bar{\lambda}\bar{z}\otimes\lambda\bar{x}\bar{z} + \bar{\lambda}\bar{z} = \bar{g}\otimes\bar{g},
\end{aligned}
$$

and, analogously, $\Delta(\bar{h}) = \bar{h}\otimes\bar{h}$. Hence, $\bar{1}, \bar{x}, \bar{g}, \bar{h}$ are distinct group-likes elements. So, by Lemma 2.1.7, they are linearly independent, we can conclude that $\overline{H}$ is in fact a group algebra $\mathbb{C}[\bar{G}]$, where $\bar{G} = \{\bar{1}, \bar{x}, \bar{g}, \bar{h}\}$. Note that each element of $\bar{G}$ has order two and, moreover, $\bar{G}$ is an abelian group. That is to say that $\bar{G}$ is the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$, i.e., $\overline{H} \cong R$. Hence the Hopf ideals of $\overline{H}$ are of the form (0), $\overline{H}(\bar{1}-\bar{x})$, $\overline{H}(\bar{1}-\bar{g})$, $\overline{H}(\bar{1}-\bar{h})$.

Now, noting that $H_8I \subseteq J$, we get that J/H_8I is a Hopf ideal of $\overline{H}$ and thus is equal to one of the Hopf ideals described above. Hence, $J = H_8I$, or $J = H_8(1-g)H_8 + H_8I$, or $J = H_8(1-h)H_8 + H_8I$, or $J = H_8(1-x)H_8 + H_8I$. The latter case cannot happen because otherwise $1-x \in I$. So the only cases are $J = H_8I$, or $J = H_8(1-(\lambda xz + \bar{\lambda}z))H_8 + H_8I$, or $J = H_8(1-(\bar{\lambda}xz + \lambda z))H_8 + H_8I$. $\square$

These chains of lemmas prove Theorem 4.3.1, i.e., we conclude that the Hopf ideals of H_8 are (0), $H_8\ker(\epsilon_{|_R})$, $\ker(\epsilon)$, $H_8\langle x-y\rangle$, $H_8\langle x-y, 1-(\lambda xz+\bar{\lambda}z)\rangle$ or $J = H_8\langle x-y, 1-(\lambda z+\bar{\lambda}xz)\rangle$.

4.4 Action on the quantum polynomial algebras

Let $M = (m_{ij}) \in M_{r\times r}(F^\times)$ be a square matrix of size r such that $m_{ii} = m_{ij}m_{ji} = 1$. Let $A_M = F_M[u_1, \ldots, u_r]$ be the *quantum polynomial algebra*, i.e., the associative $\mathbb{C}$-algebra generated by $u_1, \ldots, u_r$ subject to the relations

$$u_i u_j = m_{ij} u_j u_i, \quad 1 \le i, j \le r.$$

For more on quantum polynomial algebras, see [5] and [8, Appendix I.14 and Chapter I.2]. Alternatively, quantum polynomial algebras can be constructed as iterated Ore extension of automorphism type

$$A_M = F[u_1][u_2, \tau_2] \cdots [u_r, \tau_r],$$

where $\tau_i(u_j) = m_{ij}u_j$ for all i, j with $1 \le j < i \le r$.

Let $F = \mathbb{C}$, the field of complex numbers. In this section, we will present inner faithful actions of H_{2n^2} on A_M. Recall that the Hopf algebras H_{2n^2} constructed in Section 4.2 are of the form

$$H_{2n^2} = R[z; \sigma]/\langle z^2 - t\rangle,$$

where $R = \mathbb{C}[\mathbb{Z}_n \times \mathbb{Z}_n]$ with generators x and y. Recall also that for a nth primitive root of unity q,

$$\Delta(z) = J(z \otimes z) = \frac{1}{n}\sum_{i,j=0}^{n-1} q^{-ij} x^i z \otimes y^j z$$

Let $\tau \in S_r$ be a permutation. We define de following action of H_{2n^2} on A_M:

$$x \cdot u_i = \lambda_i u_i; \quad y \cdot u_i = \mu_i u_i; \quad z \cdot u_i = u_{\tau(i)},$$

where $\lambda_i, \mu_i \in \mathbb{C}$. In order for these relations to define an action of H_{2n^2} on A_M, the first thing to note is that since $x^n = y^n = 1$, we must have that λ_i and μ_i are roots of unity of order n, for all $i \in \{1, \ldots, r\}$. Moreover, since $zy = xz$ in H_{2n^2}, we should have that $(zy)\cdot u_i = (xz)\cdot u_i$, for all $i \in \{1, \ldots, r\}$. This happens if and only if $\mu_i = \lambda_{\tau(i)}$ for all $i \in \{1, \ldots, r\}$.

We also must have that $z \cdot (u_k u_l) = m_{kl} z \cdot (u_l u_k)$. This is equivalent to the following equivalent equations:

$$\sum_{i,j=0}^{n-1} q^{-ij} (\lambda_{\tau(k)}^j u_{\tau(k)})(\mu_{\tau(l)}^i u_{\tau(l)}) = m_{kl} \sum_{i,j=0}^{n-1} q^{-ij} (\lambda_{\tau(l)}^j u_{\tau(l)})(\mu_{\tau(k)}^i u_{\tau(k)})$$

$$m_{\tau(k)\tau(l)} \sum_{i,j=0}^{n-1} q^{-ij} (\lambda_{\tau(k)}^j \mu_{\tau(l)}^i) u_{\tau(l)} u_{\tau(k)} = m_{kl} \sum_{i,j=0}^{n-1} q^{-ij} (\lambda_{\tau(l)}^j \mu_{\tau(k)}^i) u_{\tau(l)} u_{\tau(k)}$$

$$m_{\tau(k)\tau(l)} \sum_{i,j=0}^{n-1} q^{-ij} \lambda_{\tau(k)}^j \mu_{\tau(l)}^i = m_{kl} \sum_{i,j=0}^{n-1} q^{-ij} \lambda_{\tau(l)}^j \mu_{\tau(k)}^i.$$

Writing $C(k,l) := \sum_{i,j=0}^{n-1} q^{-ij} \lambda_{\tau(k)}^j \mu_{\tau(l)}^i$, we get $z \cdot (u_k u_l) = m_{kl} z \cdot (u_l u_k) \Leftrightarrow C(k,l) m_{\tau(k)\tau(l)} = C(l,k) m_{kl}$.

Since λ_i and μ_i must be nth roots of unity for all $i \in \{1, \dots, r\}$, and q is a primitive nth root of unit, there must be $a_i, b_i \in \{0, \dots, n-1\}$ such that $\lambda_i = q^{a_i}$ and $\mu_i = q^{b_i}$. Hence, we must have

$$\begin{aligned} C(k,l) &= \sum_{i,j=0}^{n-1} q^{-ij} \lambda_{\tau(k)}^j \mu_{\tau(l)}^i \\ &= \sum_{i,j=0}^{n-1} q^{-ij} q^{a_{\tau(k)} j} q^{b_{\tau(l)} i} \\ &= \sum_{i=0}^{n-1} q^{b_{\tau(l)} i} \left(\sum_{j=0}^{n-1} q^{(a_{\tau(k)} - i) j} \right) \\ &= n q^{a_{\tau(k)} b_{\tau(l)}}. \end{aligned}$$

Therefore,

$$\begin{aligned} z \cdot (u_k u_l) = m_{kl} z \cdot (u_l u_k) &\Leftrightarrow q^{a_{\tau(k)} b_{\tau(l)}} m_{\tau(k)\tau(l)} = q^{a_{\tau(l)} b_{\tau(k)}} m_{kl} \\ &\Leftrightarrow m_{\tau(k)\tau(l)} = q^{a_{\tau(l)} b_{\tau(k)} - a_{\tau(k)} b_{\tau(l)}} m_{kl}. \end{aligned}$$

From the discussion above, we state the following lemma.

Lemma 4.4.1. *For any $n, r > 1$, primitive nth root of unity q, integers $0 \leq a_i, b_i \leq n-1$, for $i \in \{1, \dots, r\}$, permutation $\tau \in S_r$, and matrix $M = (m_{ij}) \in M_{r \times r}(\mathbb{C})$ such that $m_{ij} =$*

$m_{ij}m_{ji} = 1$ and $m_{\tau(i)\tau(j)} = q^{a_{\tau(j)}b_{\tau(i)} - a_{\tau(i)}b_{\tau(j)}}m_{ij}$, for all i, j, there exists an action of H_{2n^2} on the quantum polynomial algebra A_M with $x \cdot u_i = q^{a_i}u_i$, $y \cdot u_i = q^{b_i}u_i$ and $z \cdot u_i = u_{\tau(i)}$.

Consider the equation $m_{\tau(i)\tau(j)} = q^{a_{\tau(j)}b_{\tau(i)} - a_{\tau(i)}b_{\tau(j)}}m_{ij}$. For $k = l$, this only says that $m_{\tau(k)\tau(k)} = m_{kk}$, which has to be equal to 1. For each pair $(k, l) \in \{1 \dots, r\}^2$ denote its orbit by

$$D_{kl} = \{(\tau^s(k), \tau^s(l)) : s \in \mathbb{N}\}.$$

Fix (k, l) with $k \neq l$ and let d be the size of D_{kl}. Using d-times the relation $m_{\tau(k)\tau(l)} = q^{a_{\tau(l)}b_{\tau(k)} - a_{\tau(k)}b_{\tau(l)}}m_{kl}$, we get

$$m_{kl} = q^{\sum_{s=0}^{d-1} a_{\tau^s(l)}b_{\tau^s(k)} - a_{\tau^s(k)}b_{\tau^s(k)}}m_{kl}.$$

As m_{kl} is invertible and q is a primitive nth root of unity we have an equality in $\mathbb{Z}_n$:

$$\sum_{l=0}^{d-1} a_{\tau^l(j)}b_{\tau^l(i)} = \sum_{l=0}^{d-1} a_{\tau^l(i)}b_{\tau^l(j)}. \tag{4.11}$$

Example 4.4.2. *Let $r = 2$, and $\tau = (12)$. Then equation (4.11) says for $(i, j) = (1, 2)$:*

$$a_2b_1 + a_1b_2 = a_1b_2 + a_2b_1$$

which is always true. Nevertheless, since

$$m_{12}^{-1} = m_{21} = q^{a_2b_1 - a_1b_2}m_{12} \qquad \Rightarrow \qquad m_{12}^2 = q^{a_1b_2 - a_2b_1}.$$

Thus if $p = \sqrt{q}$ is a square root of q, then $m_{12} = p^{a_1b_2 - a_2b_1}$ is completely determined up to a sign.

Example 4.4.3. *$r = 3$, $\tau = (123)$. Then equation (4.11) says for $(k, l) = (1, 2)$:*

$$a_2b_1 + a_3b_2 + a_1b_3 = a_1b_2 + a_2b_3 + a_3b_1 \in \mathbb{Z}_n$$

In other words

$$(a_1, a_2, a_3) \cdot (b_3 - b_2, b_1 - b_3, b_2 - b_1)^t = 0.$$

Hence, for any vector $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{Z}_n^3$*, we need to find a vector* $\mathbf{c} = (c_1, c_2, c_3)$ *that is orthogonal to* $\mathbf{a}$ *and satisfies* $c_1+c_2+c_3 = 0$*, because then one can choose* $\mathbf{b} = (b, b+c_3, c_1+c_3+b)$ *for any* b*.*

For instance, if $\mathbf{a} = (1,1,1)$ *and* $\mathbf{c} = (1,1,-2)$*, then* $\mathbf{b} = (b, b-2, b-1)$ *satisfies the conditions for any* b *and hence for any* c *one could define* $m_{12} = c, m_{23} = q^2c, m_{31} = qc$*, i.e. the matrix entries are*

$$\begin{pmatrix} 1 & c & q^{-1}c^{-1} \\ c^{-1} & 1 & q^2c \\ qc & q^{-2}c^{-1} & 1 \end{pmatrix}$$

while the action is defined as $x \cdot u_i = qu_i$ *for all* $i = 1,2,3$ *and*

$$y \cdot u_1 = q^b u_1, \qquad y \cdot u_2 = q^{b-2}u_2, \qquad y \cdot u_3 = q^{b-1}u_3$$

$$z \cdot u_1 = u_2, \qquad z \cdot u_2 = u_3, \qquad z \cdot u_3 = u_1.$$

In the sequel, we present some condition to determine whether the action given by Lemma 4.4.1 is inner faithful.

Lemma 4.4.4. *Suppose that there is an action of* H_{2n^2} *on* A_M *as in Lemma 4.4.1. If for all* $i, j \in \{0, \ldots, n-1\}$*, with* $(i,j) \neq (0,0)$*, there exists* $k \in \{1, \ldots, r\}$ *such that* $ia_k + jb_k \not\equiv 0 \pmod{n}$*, then the action is inner faithful.*

Proof. Let I be a Hopf ideal of H_{2n^2} such that $I \cdot A_M = 0$. Suppose that $I \cap R \neq 0$. Since, by Lemma 2.2.13, $I \cap R$ is a Hopf ideal of the group algebra R, we can apply Lemma 3.1.7 and conclude that there exists a normal subgroup N of $\mathbb{Z}_n \times \mathbb{Z}_n$ such that $I \cap R = RF[N]^+$. Suppose $I \cap R \neq 0$, $N \neq \langle e \rangle$. Hence, there exist $i, j \in \{0, \ldots, n-1\}$, with $(i,j) \neq (0,0)$, such that $x^i y^j \in N$. This implies that $1 - x^i y^j \in I$. Thus, for all $k \in \{1, \ldots, r\}$, we have

$$0 = (1 - x^i y^j) \cdot u_k = (1 - q^{a_k i + b_k j})u_k,$$

which implies that $q^{a_k i + b_k j} = 1$ for all $k \in \{1, \ldots, r\}$, i.e., $a_k i + b_k j \equiv 0 \pmod{n}$ for all $k \in \{1, \ldots, r\}$. But this is a contradiction to the hypothesis. Therefore, $I \cap R = 0$, and thus, by Lemma 4.2.4, $I = 0$. So, the action must be inner faithful. $\square$

Now, this lemma can be viewed from a different perspective. Suppose we are in the context of Lemma 4.4.1. Consider the matrix

$$B = \begin{pmatrix} a_1 & \cdots & a_r \\ b_1 & \cdots & b_r \end{pmatrix} \in M_{2\times r}(\mathbb{Z}_n)$$

as a $\mathbb{Z}_n$-linear map from $f : \mathbb{Z}_n^2 \to \mathbb{Z}_n^r$ with $f(i,j) = (i,j)B$. Then f is injective if and only if

$$\forall (i,j) \neq (0,0),\ \ \exists 1 \leq k \leq r :\ \ ia_k \not\equiv jb_k \pmod{n}.$$

So, in order to apply Lemma 4.4.4, we just need to check when f is injective. And the following lemma gives a condition on the matrix B in order to check that f is injective.

Lemma 4.4.5. *If B has an invertible 2×2-minor, then f is injective. The converse holds if n is a prime number.*

Proof. Let $(i,j) \in \mathbb{Z}_n^2$, such that $f(i,j) = (0,\ldots,0)$. Then for any $k \neq l$ one has $ia_k + jb_k = ia_l + jb_l = 0$ in $\mathbb{Z}_n$, i.e.

$$(i,j)\begin{pmatrix} a_k & a_l \\ b_k & a_l \end{pmatrix} = (0,0).$$

If B has an invertible 2×2 minor, then $(i,j) = (0,0)$, i.e. f is injective.

If n is a prime number, then f is a linear map over the field $\mathbb{Z}_n$. Hence B must have rank 2 and therefore must contain an invertible 2×2-minor. $\square$

Example 4.4.6. *Let $n = 6$ and consider the map f associated to the matrix $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \end{pmatrix}$ then f is injective, but none of its 2×2 minors is invertible as their determinant is either $0, 2$ or 3 which are all zero divisors in $\mathbb{Z}_6$.*

An action of H_8 on A_M

The example that we will introduce in this subsection is due to Kirkman, Kuzmanovich, and Zhang [27, Example 7.5]. We will show that the action defined by them is in fact inner faithful.

Consider $r = 4$ for the quantum polynomial algebra A_M and $M = (m_{ij})$ be a matrix with elements satisfying:

$$m_{12} = m_{34}^{-1}, \quad m_{13} = m_{24}^{-1}, \quad m_{14}^2 = 1, \quad m_{23}^2 = -1.$$

Consider also $n = 2$ for the Hopf algebras H_{2n^2}. That is, we are just considering the Hopf algebra H_8. Hence, $q = -1$ here. Define the following relations:

$$\begin{array}{llll} x \cdot u_1 = u_1, & x \cdot u_2 = u_2, & x \cdot u_3 = -u_3, & x \cdot u_4 = u_4 \\ y \cdot u_1 = u_1, & y \cdot u_2 = -u_2, & y \cdot u_3 = u_3, & y \cdot u_4 = u_4 \\ z \cdot u_1 = u_4, & z \cdot u_2 = u_3, & z \cdot u_3 = u_2, & z \cdot u_4 = u_1. \end{array}$$

So, here we have $\tau = (14)(23) \in S_4$ and $a_1 = a_2 = a_4 = 0$, $a_3 = 1$, $b_1 = b_3 = b_4 = 0$, and $b_2 = 1$.

Since the matrix

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

contains the invertible minor $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, by Lemma 4.4.5, the map $f : \mathbb{Z}_2^2 \to \mathbb{Z}_2^4$ with $f(i,j) = (i,j)B$ is injective. Hence, by Lemma 4.4.4, the action is inner faithful.

Another argument to show the inner faithfulness of this is action can be done by considering the classification of the Hopf ideals of H_8 that we have done. Since $(x - y)$ belongs to all of them which are non-zero, we just note that

$$(x - y) \cdot u_2 = 2u_2 \neq 0.$$

Therefore, the only Hopf ideal in $\mathrm{Ann}_{H_8}(A_M)$ is the zero one. Hence, the action is inner faithful.

4.5 Actions on the quantum plane

For a moment, let F be an arbitrary field. Let $0 \neq p \in F$ and consider the matrix

$$M = \begin{pmatrix} 1 & p^{-1} \\ p & 1 \end{pmatrix} \in M_{2\times 2}(F)$$

The *quantum plane* is the quantum polynomial algebra $A_M = F_M[u, v]$ with two generators. We denote the quantum plane $F_M[u, v]$ by $F_p[u, v]$ and we call p a *parameter*. When $p \neq 1$ the algebra $F_p[u, v]$ is non-commutative.

In this section we shall use what we have done in the last section in order to investigate inner faithful actions of H_{2n^2} on the quantum plane. For the case $n = 2$, we shall classify the actions under a certain condition.

4.5.1 An inner faithful action of H_{2n^2} on the quantum plane

Consider the Hopf algebras H_{2n^2} as constructed in Section 4.2. For each n, these Hopf algebras act on $A = \mathbb{C}_p[u, v]$ with $p^2 = q$, where q is the primitive nth root of unity used to construct H_{2n^2}. The action is given by:

$$\begin{aligned} x \cdot u &= qu, & y \cdot u &= u, & z \cdot u &= v, \\ x \cdot v &= v, & y \cdot v &= qv, & z \cdot v &= u, \end{aligned}$$

which corresponds to $\tau = (12) \in S_2$ and the matrix

$$B = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_{2\times 2}(\mathbb{Z}_2)$$

in the Section 4.4. By Example 4.4.2, condition (4.11) is satisfied, i.e., H_{2n^2} acts on A. Furthermore, since the matrix B is invertible in $\mathbb{Z}_2$, by Lemma 4.4.4, this action is inner faithful.

4.5.2 Actions of H_8 on the quantum plane

Let $F = \mathbb{C}$ and $A = \mathbb{C}_p[u, v]$ be the quantum plane with parameter $p \in \mathbb{C}^\times$, i.e., $vu = puv$. Let H_8 be the Hopf algebra as presented in Section 4.3. Recall that $z \in H_8$ is such that $\Delta(z) = J(z \otimes z)$ where $J = \frac{1}{2}(1 \otimes 1 + x \otimes 1 + 1 \otimes y - x \otimes y)$, and x, y are a pair of generators of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

In the following theorem we classify the possibles inner faithful actions of H_8 on A under a certain assumption.

Theorem 4.5.1. *Let $1 \neq p \in \mathbb{C}^\times$. If there is a Hopf action of H_8 on the quantum plane $A = \mathbb{C}_p[u, v]$ such that $z \cdot u = v$ and $z \cdot v = u$, then this action is inner faithful and $p^2 = -1$.*

Proof. If there is an action of H_8 on A, since x and y are group-like elements, they act as automorphisms of A (Remark 2.3.2). Hence there exist $\alpha, \beta \in \mathrm{Aut}(A)$ such that $x \cdot a = \alpha(a)$ and $y \cdot a = \beta(a)$ for all $a \in A$. Also, since $x^2 = 1 = y^2$, $\alpha^2 = id = \beta^2$.

Under the assumption that z acts by interchanging u and v, we must have $z\cdot(vu) = pz\cdot(uv)$ or equivalently

$$uv + \alpha(u)v + u\beta(v) - \alpha(u)\beta(v) = p(vu + \alpha(v)u + v\beta(u) - \alpha(v)\beta(u)). \tag{4.12}$$

Moreover, since $xz = zy$, it follows that $(xz) \cdot u = (zy) \cdot u$ and $(xz) \cdot v = (zy) \cdot v$, which implies that

$$\alpha(u) = z \cdot \beta(v) \quad \text{and} \quad \alpha(v) = z \cdot \beta(u). \tag{4.13}$$

Now, we separate the proof in cases.

CASE I: $p \neq -1$: In this case Alev and Chamarie showed in [2, 1.4.4] that the automorphisms of A are given by a torus action, i.e., x and y act as scalars on u and v. Hence we are in the situation of Section 4.4. Suppose

$$x \cdot u = (-1)^{a_1}u, \qquad x \cdot v = (-1)^{a_2}v, \qquad y \cdot u = (-1)^{b_1}u, \qquad y \cdot v = (-1)^{b_2}v,$$

and let $B = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \in M_{2\times 2}(\mathbb{Z}_2)$. By Lemma 4.4.1 and Example 4.4.2, H acts on A if and only if

$$p^2 = (-1)^{a_1 b_2 - a_2 b_1} = (-1)^{\det(B)}.$$

Since $p \neq -1$, we must have $\det(B) = 1$ and so $p^2 = -1$. Hence, by Lemma 4.4.5 the action is inner faithful.

CASE II: $p = -1$**:** In this case Alev and Chamarie showed in [2, 1.4.4] that $\mathrm{Aut}(A)$ is a semidirect product of $(\mathbb{C}^\times)^2$ with the cyclic group of order 2 given by the automorphism τ that flips u and v. Hence any automorphism of A is either an element of $(\mathbb{C}^\times)^2$ or a product of an element $(\alpha_0, \alpha_1) \in (\mathbb{C}^\times)^2$ and τ. By 4.13, if β is just given by a torus action, then α has to be also given by a torus action, and if β is given by a torus action and τ, then α has to be also given by a torus action and τ. Therefore, these are the only two possible cases for α and β.

CASE II.a: If β is given only by a torus action, then we are in the same situation as CASE I and hence, by what we have done for CASE I, necessarily $p^2 = -1$, which contradicts $p = -1$. So, β cannot be given by a torus action.

CASE II.b: Suppose that both α and β are compositions of a torus action and τ, then there are $(\alpha_0, \alpha_1), (\beta_0, \beta_1) \in (\mathbb{C}^\times)^2$ such that

$$x \cdot u = \alpha(u) = \alpha_0 v, \qquad x \cdot v = \alpha(v) = \alpha_1 u, \qquad y \cdot u = \beta(u) = \beta_0 v, \qquad y \cdot v = \beta(v) = \beta_1 u.$$

Then equation (4.12) yields

$$uv + \alpha_0 v^2 + \beta_1 u^2 + \beta_1 \alpha_0 uv = uv - \alpha_1 u^2 - \beta_0 v^2 + \beta_0 \alpha_1 uv,$$

which is equivalent to

$$(\alpha_1 + \beta_1)u^2 + (\alpha_0 + \beta_0)v^2 + (\beta_1 \alpha_0 - \beta_0 \alpha_1)uv = 0$$

and implies $\beta_i = -\alpha_i$, for $i = 0, 1$, and $\beta_1 \alpha_0 = \beta_0 \alpha_1$. Now, since $\alpha^2 = id = \beta^2$, we must have that $\alpha_0 \alpha_1 = 1$ and $\beta_0 \beta_1 = 1$. Also, by 4.13, we must have that $\alpha_1 = \beta_0$ and $\alpha_0 = \beta_1$. Hence, $\alpha_0^2 = -\alpha_0 \alpha_1 = -1$. And so $\alpha_0 = \pm i$. Therefore, for $\alpha_0 \in \{-i, i\}$, the options are $\beta_1 = \alpha_0$, $\beta_0 = -\alpha_0$, $\alpha_1 = -\alpha_0$. But note that

$$xz \cdot (vu) = (vu - \alpha_0 u^2 - \alpha_0 v^2 - vu) \quad \text{and} \quad zy \cdot (vu) = (-uv - \alpha_0 u^2 - \alpha_0 v^2 - uv).$$

This leads to a contradiction, since $xz = yz$. Therefore, there can be no action at all for the case where $p = -1$. □

Remark 4.5.2. *If $p = 1$, then $A = \mathbb{C}[u, v]$ is the commutative polynomial ring, which is a commutative domain. Therefore, Etingof and Walton's result [18, Theorem 1.3] guarantee that there cannot be any inner faithful action of H_{2n^2} on A, since it is not a group algebra.*

The following example is due to Kirkman, Kuzmanovich, and Zhang [27].

Example 4.5.3 ([27, Example 7.4])**.** *Let $A = \mathbb{C}_p[u, v]$ be the quantum plane with parameter p with $p^2 = -1$. Then H_8 acts on A as follow:*

$$x \cdot u = -u, \qquad y \cdot u = u, \qquad z \cdot u = v,$$

$$x \cdot v = v, \qquad y \cdot v = -v, \qquad z \cdot v = u.$$

Here we are in CASE I as in the proof of Theorem 4.5.1. Hence, by the same Theorem, this action is inner faithful.

Theorem 4.5.1 says that it is not possible to define an inner faithful action on $\mathbb{C}_{-1}[u, v]$ for z interchanging u and v. But when that is not the case, it is possible to define an inner faithful Hopf action as in the following example, which is also due to Kirkman, Kuzmanovich, and Zhang [27].

Example 4.5.4 ([27, Example 7.6])**.** *Let $A = \mathbb{C}_{-1}[u, v]$ be the quantum plane. Then H_8 acts on A as follow:*

$$x \cdot u = u, \qquad y \cdot u = -u, \qquad z \cdot u = u,$$

$$x \cdot v = v, \qquad y \cdot v = -v, \qquad z \cdot v = -v.$$

We claim that this action is inner faithful. Recall the classification of the Hopf ideals of H_8 that we have done in 4.3.2. Note that $(x - y) \cdot u = 2u \neq 0$ and then, since $(x - y)$ belongs to all Hopf ideals of H_8, it follows that the action is inner faithful.

Also, due to Allman [3], where all the details are checked, we present another inner faithful actions of H_8 on A when the action of z is not given simply by interchanging u and v.

Example 4.5.5 ([3, Example 4.2.6])**.** *Let* $A = \mathbb{C}_{-1}[u, v]$ *be the quantum plane. Then* H_8 *acts on* A *as follow:*

$$x \cdot u = v, \qquad y \cdot u = -v, \qquad z \cdot u = iv,$$
$$x \cdot v = u, \qquad y \cdot v = -u, \qquad z \cdot v = iu.$$

We claim that this action is inner faithful. The argument is the same we have applied in Example 4.5.4, i.e., $(x - y) \cdot u = 2v \neq 0$ *and then, since* $(x - y)$ *belongs to all Hopf ideals of* H_8*, it follows that the action is inner faithful.*

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